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A NEW INTEGER SEQUENCE BASED ON THE SUM OF THE DIGITS OF INTEGERS

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Abstract: A new integer sequence defined on the basis of the EP numbers is introduced (A117570). The EP numbers are defined as $EP(x) = (Ab^n + Bb^{n-1} + Cb^{n-2} + \cdots + Pb + Q)(A + B + C + \cdots + P + Q)$, where $x = Ab^n + Bb^{n-1} + Cb^{n-2} + \cdots + Pb + Q$, $n, A, B, C, \cdots, Q$ are non-negative integers and b is the base. We illustrate some general characteristics of EP numbers and the EP sequence. We observe the existence of pairs of consecutive EP numbers (twin EP numbers) as well as the existence of EP numbers that can be built by different ways. The EP sequence appears to be uniformly distributed and the plot of x vs. $EP(x)$ displays a characteristic piecewise parabolic shape.

INTRODUCTION

Since the old times of the Pythagoreans numbers, in particular integers, have always had a magic touch (WELLS, 1986; POGLIANI, 2006). Kronecker said that "*God himself made the integers: everything else is the work of man*" (WELLS, 1986). Another captivating area of research is that related to integer sequences. Integer sequences appear in many different areas of scientific research, such as number theory, combinatorics, graph theory, game theory, physics, chemistry, computer sciences, communications, and so forth (SLOANE, 1999). In addition, it is also of interest in recreational and educational mathematics (BEILER, 1964). As Gauss remarked "in arithmetic the most elegant theorems frequently arise experimentally as the result of a more or less unexpected stroke of good fortune" (WELLS, 1986). The current work has been inspired by such a kind of "unexpected stroke of good fortune" despite it is still distant from being among "the most elegant theorems".

The current report is motivated by the New Year 2008, which is just starting. The number 2008 is a sacred number in Hindu tradition together with 108 and 1008. We accidentally discovered that the number 2008 can be expressed as the product of a number and the sum of the digits of such number, e.g., $2008 = 251(2+5+1)$. Then, we proceed by proposing a sequence of integers formed by applying this operation to integer numbers. Our objective in this short note is to motivate others to the deepest study of this sequence, its properties and possible applications. Other similar sequences, such as the Kaprekar numbers (KAPREKAR, 1980-81) (A006886) (SLOANE and PLOUFFE, 1995) has attracted such attention (IANNUCCI, 2000) and the Fibonacci numbers (A000045) (SLOANE and PLOUFFE, 1995) have been recently used to design robust networks with desired properties (ESTRADA, 2007).

BUILDING METHOD

To start with we will define the process through which we generate the numbers of the sequence, which hereafter will be named as EP numbers. EP is simply the first initial of both authors.

Definition 1: Let X be a non-negative integer. X is an EP number for the base b if there exist non-negative integers $n, A, B, C, \cdots, Q$ satisfying

$$X = \left(Ab^n + Bb^{n-1} + Cb^{n-2} + \cdots + Pb + Q\right)\left(A + B + C + \cdots + P + Q\right) \qquad (1)$$

Then, if $x = Ab^n + Bb^{n-1} + Cb^{n-2} + \cdots + Pb + Q$ the corresponding $EP(x)$ number can be generated using the expression (1). It is straightforward to realize that the numbers X define an integer sequence. The first numbers of this sequence for the base $b = 10$ are given below:

0,1,4,9,10,16,22,25,36,40,49,52,63,64,70,81,88,90,100,112,115,124,136,144,160,162,175,
190,198,202,205,208,220,238,243,250,252,280,301,306,319,324,333,352,360,364,370,400
,405,412,418,424,427,448,460,468,484,486,490,496,517,520,550,565,567,568,576,603,61
0,616,630,637,640,648,655,684,700,715,729,730,738,742,754,792,805,808,810,814,820…

The EP sequence has been included in the *Encyclopedia of Integer Sequences* with code (A117570) (Sloane and Plouffe, 1995). The following describe a method to find the X_{k+1} th EP number in the sequence by knowing the k th one.

Lemma: Let X_k be the k th EP number in the series defined by (1). Then, the next EP number X_{k+1} is given by,

$$X_{k+1} = X_k + A\left(b^n + 1\right) + B\left(b^{n-1} + 1\right) + \cdots + 2Q + 1 \qquad (2)$$

A PHYSICAL MEANING FOR THE EP NUMBERS

In order to understand the physical meaning of the EP numbers we consider a set S of cardinality $x = |S|$. For the sake of simplicity we consider the elements of the set S as point particles. Let us consider a physical process that group these point particles in r subsets $s_1, s_2, \cdots, s_r$, such that $S = \bigcup s_i$. Let us consider that the cardinalities of these subsets are $b^n = |s_1|, b^{n-1} = |s_2|, \cdots, b = |s_{r-1}|, b^0 = |s_r|$, such that

$$|S| = A|s_1| + B|s_2| + \cdots + P|s_{r-1}| + Q, \qquad (3)$$

where $A, B, \cdots, P, Q$ are the number of subsets $s_1, s_2, \cdots, s_r$, respectively and b is the base of the process.

For instance, if we consider 12 point particles and a base $b = 10$ we have a "physical" process that group these particles in three groups, one having 10 particles and two having one particle each. If we consider 325 particles then the total number of groups is 12. That is, there are 3 groups of 10^2 particles, 2 groups of 10 particles and 5 groups having one particle each. Then, it is obvious that the total number of groups formed by such kind of process is given by,

$$c(x) = A + B + C + \cdots + P + Q \,. \tag{4}$$

Now, let us consider a process generating $c(x)$ groups for a base b. The particle i can be in any of these $c(x)$ groups. Let us call "a scenario" to the situation in which the ith particle is localized on a given group. For instance, in the example of the 12 particles one scenario consists in the situation in which the particle 1 is locate in the group having other 9 particles. It is evident that every particle could be in $c(x)$ different scenarios. Thus, the total number of scenarios is equal to the number of particles multiplied by the number of groups,

$$\varepsilon(x) = xc(x)$$

which by substitution of the expressions for x and $c(x)$ immediately gives the expression for the EP numbers,

$$\varepsilon(x) = \left(Ab^n + Bb^{n-1} + Cb^{n-2} + \cdots + Pb + Q\right)\left(A + B + C + \cdots + P + Q\right) = EP(x) \tag{5}$$

Consequently, the EP numbers represent the number of scenarios that can be formed by grouping x particles in $c(x)$ groups using a base b.

SOME PROPERTIES OF THE EP SEQUENCE

Here we are going to show some observational properties of the EP sequence that might be of interest. To start with we define EP numbers of different orders.

Definition 2: Let X be an EP number on the base b. Then, X is an EP number of order q if it can be obtained by q different expressions of type (1).

For instance, for $b = 10$,

$112 = 16(1+6)$	First order EP number
$36 = 06(0+6)$ $36 = 12(1+2)$	Second order EP number
$900 = 150(1+5+0)$ $900 = 300(3+0+0)$ $900 = 75(7+5)$	Third order EP number

Another interesting observation that we have carried out in the EP sequence is the existence of pairs of consecutive EP numbers whose difference is exactly one.

Definition 3: Let X_k and X_{k+1} be two consecutive EP numbers. Then, if $X_{k+1} = X_k + 1$ the numbers are named *twin EP numbers*.

For instance, the following are examples of twin EP numbers,

$$63 = 21(2+1)$$
$$64 = 08(0+8)$$

$$567 = 63(6+3)$$
$$568 = 71(7+1)$$

$$1215 = 135(1+3+5)$$
$$1216 = 152(1+5+2)$$

A plot of the integers versus the corresponding EP numbers, i.e., x vs. $EP(x)$ is illustrated below for natural numbers between 0 and 1000 (top) and a magnification for the numbers between 0 and 100 (bottom).

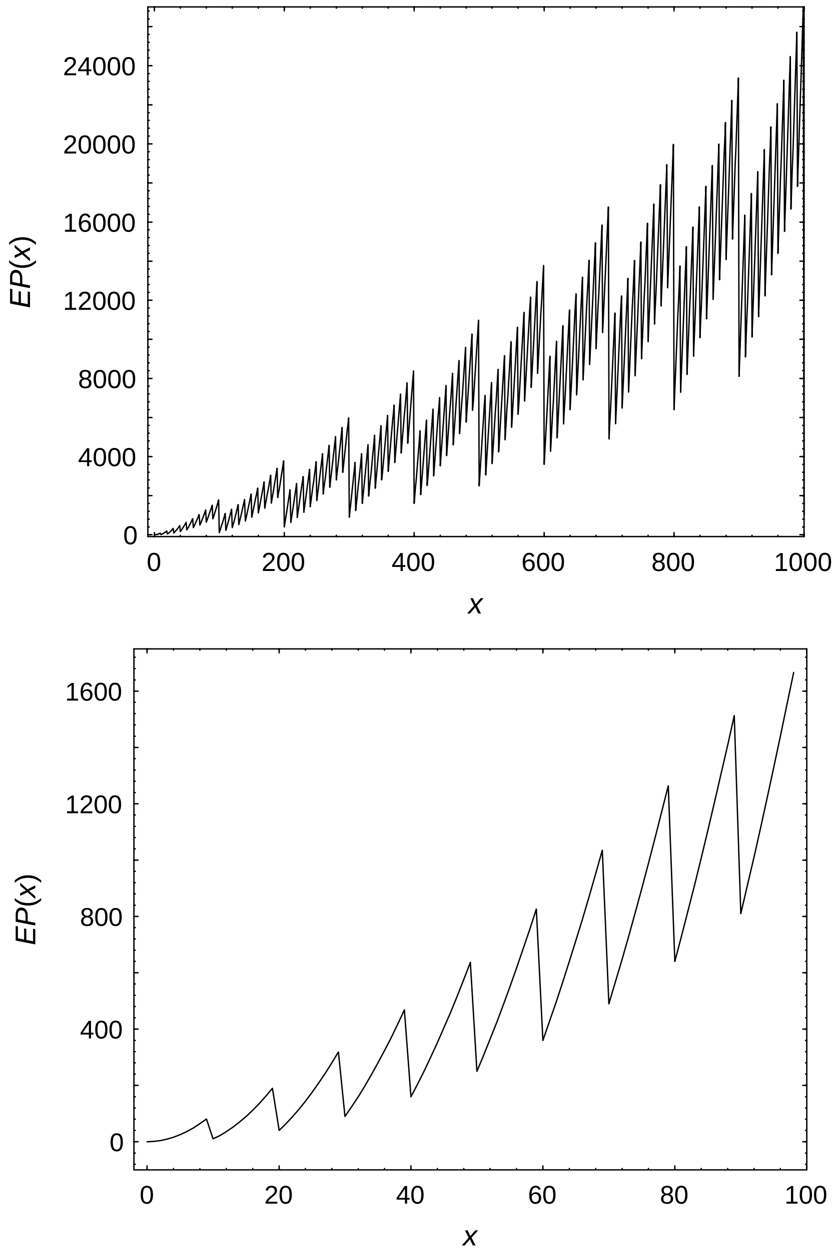

Figure 1. Plots of x vs. $EP(x)$ for natural numbers between 0 and 1000 (top) and a magnification for the numbers between 0 and 100 (bottom).

As can be seen in these plots $EP(x)$ can be expressed as a piecewise function of x^2. In general, $EP(x)$ can be expressed as

$$EP(x) = x^2 + A^2 b^n \left(1 - b^n\right) + B^2 b^{n-1}\left(1 - b^{n-1}\right) + C^2 b^{n-2}\left(1 - b^{n-2}\right) + \cdots + ABb^n\left(1 + b^{-1} - 2b^{n-1}\right)$$
$$+ ACb^n\left(1 + b^{-2} - 2b^{n-2}\right) + BCb^{n-1}\left(1 + b^{-1} - 2b^{n-2}\right) + \cdots + AQ\left(1 - b^n\right) + BQ\left(1 - b^{n-1}\right)$$
$$+ CQ\left(1 - b^{n-2}\right) + \cdots + PQ(1 - b) \qquad (6)$$

Then, this formula can be transformed to a second degree polynomial by taking the specific values of the parameters $A, B, C, \cdots, Q$. For instance, for the particular case of numbers of the type $x = 100A + 10B + C$ we can obtain,

$$EP(x) = \frac{1}{10}x^2 - \frac{9(10A - C)}{10}x \qquad (7)$$

which exactly reproduce the EP numbers between 0 and 999 displayed in the plots of Figure 2.

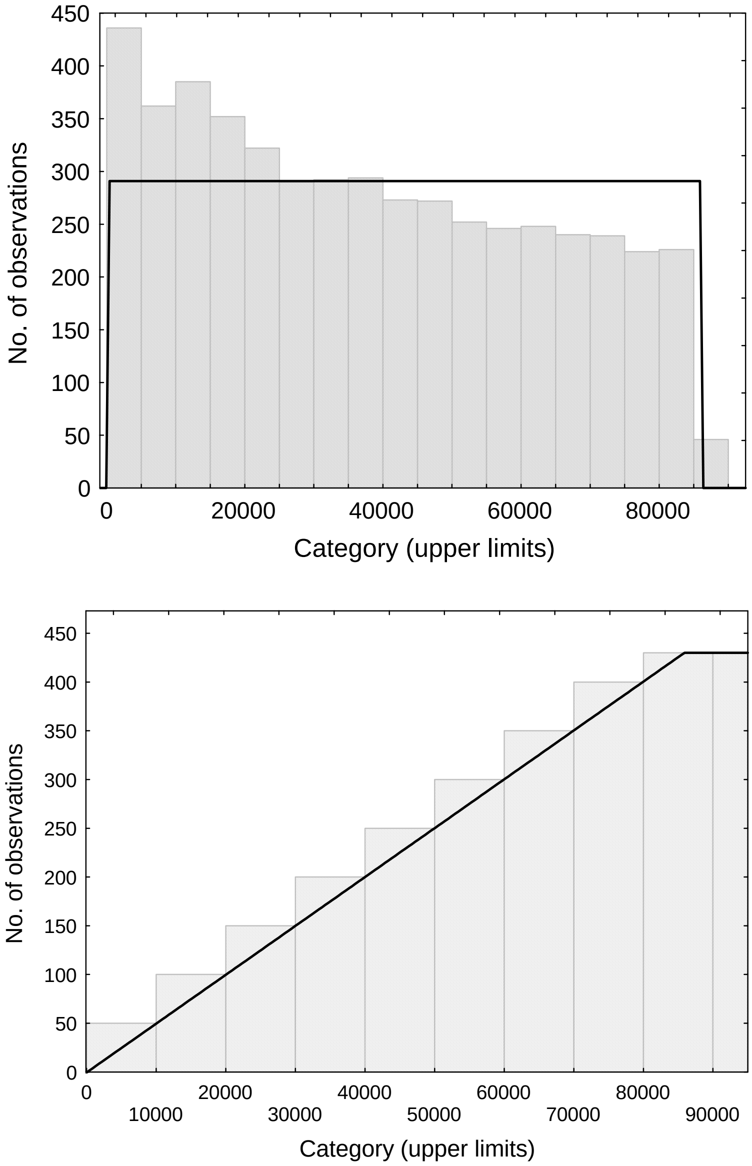

Figure 2. Distribution (top) and cumulative distribution (bottom) of the first 5000 EP numbers. The solid lines represent the predicted ideal distribution.

Finally, we investigate the distribution of the first 5000 EP numbers. As can be seen in the Figure 2 the EP numbers appear to be distributed according to a uniform (square) distribution. This indicates that for each member of the EP sequence, all intervals of the same length on the distribution are equally probable. In other words, the probability of finding EP numbers in different intervals of the same length is the same.

In summary, we introduce a new series of numbers which are created as the product of an integer and the sum of its digits. We have shown how these numbers can be obtained by knowing the preceding element in the sequence and we have illustrated some observational characteristics of these numbers and its sequence. An important remaining motivation for extending this work is the following, **Open question:** How to know whether a given integer number is an EP number? We hope that the current work motivates new research for the use of EP numbers and its sequence in different areas, such as cryptographic analysis.

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