Linear Algebra and its Applications 430 (2009) 1886–1896

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Returnability in complex directed networks (digraphs)

Ernesto Estrada [a,b,c,d,*], Naomichi Hatano [d]

[a] Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, UK
[b] Department of Physics, University of Strathclyde, Glasgow G1 1XH, UK
[c] Institute of Complex Systems at Strathclyde, University of Strathclyde, Glasgow G1 1XH, UK
[d] Institute of Industrial Science, University of Tokyo, Komaba, Meguro 153-8505, Japan

A R T I C L E I N F O

Article history:
Received 12 September 2008
Accepted 22 September 2008
Available online 31 December 2008

Submitted by R.A. Brualdi

AMS classification:
05C20
05C50
05C90
05C75
15A18

Keywords:
Graph spectrum
Digraphs
Estrada index
Reciprocity
Complex networks

A B S T R A C T

The concept of returnability is proposed for complex directed networks (digraphs). It can be seen as a generalization of the concept of reciprocity. Two measures of the returnability are introduced. We establish closed formulas for the calculation of the returnability measures, which are also related to the digraph spectrum. The two measures are calculated for simple examples of digraphs as well as for real-world complex directed networks and are compared with the reciprocity.

1. Introduction

The study of complex networks has become an important area of cross-disciplinary research [1,2]. Many real-world complex networks are directional, indicating the asymmetric relationship between the entities [1–5]. For instance, many chemical and biochemical reactions in metabolic networks are unidirectional [6–8]. In ecological food webs, some species predate others in a unidirectional way [9–11] and in social and economical networks, some of the relationships in one direction do not necessarily

* Corresponding author. Address: Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, UK.
E-mail addresses: ee@math.strath.ac.uk, estrada66@yahoo.com (E. Estrada).

0024-3795/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2008.09.033

mean that the reverse relation is also present in the network [12,13]. In the Internet at the autonomous system (AS) level, a national AS, for example, that provides services to a regional AS, which at the same time provides services to a local AS, cannot be also the customer of the local AS [14,15]. This last case is very illustrative of two phenomena, the directionality of relationships and the lack of returnable cycles in a complex network. The study of directed complex networks have only been put forward in recent years as an important direction of research in this field. Despite its evident importance, most of the work in the modelling literature is still concerned with undirected networks.

One property of directed networks that has received recent attention is the reciprocity r, which is the proportion of directed links to the total number of links in the network [12]. Complex networks have been shown to display the non-random presence of reciprocity [16]. It has also been observed recently that even a small change in reciprocity can change dramatically the properties of percolation in complex networks [17]. Moreover, reciprocity has been recognised as a relevant parameter for understanding the topology or functionality of certain complex networks [18,19]. The reciprocity has been studied in physics contexts as the multispecies grand-canonical models [20] and networks with degree correlations [21]. However, this measure has never been analysed in a wide graph–theoretic context. Here we propose to redefine this graph invariant in terms of graph spectral parameters and then generalize it to a wider context of graph-theoretic measures for digraphs. In doing so we extend here the definition of the *subgraph centrality* index [22] to complex directed networks.

The subgraph centrality was proposed by one of the present authors (E.E.) to account for the participation of a node in all subgraphs of the network, giving higher weights to the smaller subgraphs [22]. It is also known in the literature as the Estrada index [23–26] of a graph, and has found several applications for the characterization of the topology of complex networks [27–29].

2. From reciprocity to returnability of directed graphs

We represent by $G = (V, E)$ an undirected network, where V is the set of nodes and E is the set of links. A directed network with no loops, or digraph, D, consists of a finite set of nodes V of cardinality $\#|V| = N$ and a set of *ordered* pairs of distinct nodes from V, which are called arcs or directed links. The arc pq goes from p to q. A (directed) walk of length l is any sequence of (not necessarily different) vertices $v_1, v_2, \dots, v_l, v_{l+1}$ such that for each $i = 1, 2, \dots, l$ there is an arc from v_i to v_{i+1}. A closed walk (CW) of length l is a walk in which $v_{l+1} = v_1$ [30]. A path is a walk in which all nodes are distinct and a *cycle* is a non-trivial closed walk with all nodes (except the first and last) distinct. A symmetric pair of arcs is the pair pq and qp. A digraph is *symmetric* if every arc pq in the digraph has its symmetric partner qp. If the presence of pq in a digraph excludes the presence of qp the graph is called asymmetric [31]. The underlying graph $G(D)$ of the digraph D is the graph that results from replacing each directed arc with an undirected edge [31].

The network reciprocity is defined as

$$r = \frac{L^{\leftrightarrow}}{L},$$

where $L^{\leftrightarrow}$ is the number of symmetric pairs of arcs, and L is the total number of links [12]. Thus, if there is a link pointing from A to B, the reciprocity measures the probability that there is also a link pointing from B to A.

If we consider a particle moving between the nodes of the network, the reciprocity can be interpreted as the probability that such a particle returns to the starting node after visiting any of its nearest neighbours, i.e., after completing a closed walk of length two. The following is a very well-known result in graph theory:

Theorem 1 [30]. *Let* $\mathbf{A}$ *be the adjacency matrix of a graph. The quantity* $\mu_k(i,j) = (\mathbf{A}^k)_{ij}$ *counts the number of different walks (for* $i \neq j$*) or closed walks (for* $i = j$*) of length* k *between nodes* i *and* j.

Then, it is straightforward to realize that the reciprocity of a network is equal to the number of closed walks of length $k = 2$ divided by the number of closed walks of length $k = 2$ in the underlying graph of the digraph,

$$r = \frac{\mu_2(D)}{\mu_2[G(D)]} = \frac{\mathrm{tr}\mathbf{D}^k}{\mathrm{tr}\mathbf{A}^k}, \tag{1}$$

where **D** and **A** are the adjacency matrices of the digraph and its underlying graph, respectively.

Then, the reciprocity can be seen in a more general context. In such a context we can consider the existence of a walk of length k from A to B and ask for the probability that there will be also a walk of length l (not necessarily different from k) from B to A. Of course, when $k = l = 2$ we have the reciprocity as defined previously. Shorter walks, however, are typically more important than longer walks, so it is intuitively reasonable to form a length-based weighted average for accounting such probability. Taking these two conditions into account, we define the following measure for complex networks.

Definition 1. The returnability is the fraction of closed walks (returnable walks) in the digraph to the number of such walks in the underlying graph of the digraph in such a way that the closed walks are weighted in decreasing order of their lengths.

Mathematically, the returnability can be defined as

$$\chi(c_k, D) = \frac{c_2\mu_2(D) + c_3\mu_3(D) + \cdots + c_k\mu_k(D) + \cdots}{c_2\mu_2[G(D)] + c_3\mu_3[G(D)] + c_k\mu_k[G(D)] + \cdots}. \tag{2}$$

Note that we are considering digraphs and graphs without multiple links or self-loops and hence we have taken advantage of the fact that $\mu_1(D) = \mu_1[G(D)] = 0$ due to the absence of self-loops, i.e., links starting and ending at the same node. Then, it is straightforward to realize that the reciprocity is a particular case of the returnability with $c_k = 0$ for $k > 2$.

3. Quantitative measures of returnability

The result, given here without further proof, is a quantitative measure for the returnability of a digraph. We obtained the following result.

Proposition 2. *Let D be a non-empty digraph of order N. Let* **D** *be the adjacency matrix of the digraph and let* **A** *be the adjacency matrix of the underlying graph of D, namely $G(D)$. Then, if we select $c_k = 1/k!$, the returnability may be expressed as follows*:

$$\chi(c_k = 1/k!, D) = \frac{\mathrm{tr}e^{\mathbf{D}} - N}{\mathrm{tr}e^{\mathbf{A}} - N}, \tag{3}$$

where $e^{\mathbf{M}}$ represents the exponential matrix and tr *represents the trace.*

Remark 1. A variation of the previous definition of returnability can be obtained by considering only the odd (or even) closed walks in the digraph and the underlying graph, which are, respectively:

$$\chi(c_k = 1/(2k+1)!, D) = \frac{\mathrm{tr}[\sinh(\mathbf{D})] - N}{\mathrm{tr}[\sinh(\mathbf{A})] - N},$$

$$\chi(c_k = 1/(2k)!, D) = \frac{\mathrm{tr}[\cosh(\mathbf{D})] - N}{\mathrm{tr}[\cosh(\mathbf{A})] - N}.$$

If we compare the expressions for the reciprocity (1) and that for the returnability using $c_k = 1/k!$, we can see that the returnability χ gives a weight to the closed walks of length 3 which is only a third of that given to the closed walks of length 2, i.e., $c_2 = 1/2$ and $c_3 = 1/6$, whereas the reciprocity r gives a weight of zero to the closed walks of length larger than 2, $c_2 = 1$ and $c_3 = 0$.

Another intuitive choice of the weighting coefficients is to define the returnability in such a way that the longer closed walks are penalized in a more dramatic way than the shorter ones. This is introduced in the following result.

Proposition 3. *Let D be a non-empty digraph of order N. Let* **D** *be the adjacency matrix of the digraph and let* **A** *be the adjacency matrix of the underlying graph of D. Then, if we select $c_k = N^{-k}$ the returnability may be expressed as follows*:

$$\chi(c_k = N^{-k}, D) = \frac{\mathrm{tr}[(\mathbf{I} - \mathbf{D}/N)^{-1}] - N}{\mathrm{tr}[(\mathbf{I} - \mathbf{A}/N)^{-1}] - N}. \tag{4}$$

Proof. First we show that for $k < e^{\ln N+1}$ the function N^{-k} penalizes the closed walks more than $1/k!$, i.e., $N^{-k}\mu_k < (1/k!)\mu_k$, which is a condition for the new function we are looking for. Now, let $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_N$ be a non-increasing order of the eigenvalues of the matrix $\mathbf{M} = \mathbf{A}/N$ for any graph of order N. Then, it is easy to see that the maximum value for an eigenvalue of $\mathbf{M}$ is $\alpha_1 = (N-1)/N$, which is attained for the Perron–Frobenius eigenvalue of the complete graph K_N. Consequently, $|\alpha_i| < 1 \quad i = 1, \ldots, N$. This condition assures the convergence $\|\mathbf{M}^k\| \to 0$ as $k \to \infty$ for any measure $\|\cdot\|$ defined for matrix $\mathbf{M}$. It is known [32] that if $|\alpha_i| < 1$ for $i = 1, \ldots, N$, the series $\sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{N^k}$ converges to $(\mathbf{I} - \mathbf{A}/N)^{-1}$ as $k \to \infty$, which proves the result.

Incidentally, the diagonal entries of the matrix $(\mathbf{I} - \mathbf{A}/N)^{-1}$ has been proposed by Estrada and Higham [33] as a variation of the subgraph centrality [22] of undirected graphs.

An important feature of the newly introduced measures of returnability is that they can be easily compared between themselves and with the reciprocity. As we already showed in Eq. (1), the reciprocity can be expressed as the fraction of the second spectral moment of the adjacency matrices of the digraph and its underlying graph. Then, we can write a general expression for the returnability in the following form:

$$\chi(c'_k, G) = \frac{\mu_2(D) + \sum_{k=3}^{\infty} c'_k \mu_k(D)}{\mu_2[G(D)] + \sum_{k=3}^{\infty} c'_k \mu_k[G(D)]}. \tag{5}$$

Then, the reciprocity is the particular case where $c'_k = 0$ and the two reciprocity measures (3) and (4) correspond to $c'_k = k!/2$ and $c'_k = N^{k-2}$, respectively.

We now relate the measures of returnability to the graph spectra through the following result. □

Proposition 4. *Let D be a non-empty digraph of order N. Let $\sigma_1, \sigma_2, \ldots, \sigma_N$ be the eigenvalues of the adjacency matrix $\mathbf{D}$ of the digraph and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the eigenvalues of the adjacency matrix $\mathbf{A}$ of the underlying graph of D. Then, the returnability of D based on the weights $c_k = 1/k!$ and $c_k = 1/N^k$ may be expressed as follows, respectively*:

$$\chi(c_k = 1/k!, D) = \frac{\sum_{j=1}^{N}(e^{\sigma_j} - 1)}{\sum_{j=1}^{N}(e^{\lambda_j} - 1)} = \frac{\sum_{j=1}^{N} e^{\sigma_j} - N}{\sum_{j=1}^{N} e^{\lambda_j} - N}, \tag{6}$$

$$\chi(c_k = N^{-k}, D) = \frac{\sum_{j=1}^{N}\left(1 - \frac{\sigma_j}{N}\right)^{-1} - N}{\sum_{j=1}^{N}\left(1 - \frac{\lambda_j}{N}\right)^{-1} - N} = \frac{\sum_{j=1}^{N} \frac{N}{N-\sigma_j} - N}{\sum_{j=1}^{N} \frac{N}{N-\lambda_j} - N}. \tag{7}$$

Proof. It is well-known that the number of closed walks of length k in a digraph is equal to the kth spectral moment, since $\sum_{i=1}^{N}(\mathbf{D}^k)_{ii} = \mathrm{tr}\mathbf{D}^k = \sum_{i=1}^{N}\sigma_i^k$ [30]. Then,

$$\sum_{k=0}^{\infty} \mathrm{tr}\left(\frac{\mathbf{M}^k}{N^k}\right) = \sum_{k=0}^{\infty}\sum_{j=1}^{N} \frac{\beta_j^k}{N^k} = \sum_{j=1}^{N}(1 - \beta_j/N)^{-1},$$

where $\mathbf{M}$ and β_j represent the adjacency matrix and eigenvectors of the digraph or graph, respectively. The result follows immediately from the previous expression. □

Remark 2. It is straightforward to realize that the reciprocity can be expressed in terms of the eigenvalues of the digraph and its underlying graph as follows:

$$r = \sum_{j=1}^{N} (\sigma_j)^2 \left[\sum_{j=1}^{N} (\lambda_j)^2 \right]^{-1}. \tag{8}$$

Incidentally, in a previous work one of the present authors (EE) defined the *subgraph centrality* of an undirected graph as [22]

$$EE(G) = \sum_{k=0}^{\infty} \frac{\mathrm{tr}\mathbf{A}^k}{k!} = \mathrm{tr}(e^{\mathbf{A}}) = \sum_{j=1}^{N} e^{\lambda_j}. \tag{9}$$

This graph-theoretic invariant has been renamed by Gutman et al. [23–25] as the Estrada index of a graph. Several of the mathematical properties of the Estrada index as well as its relationship with graph energy and other graph parameters have been studied in the recent literature [23–29].

By analogy with (7) we can define the Estrada index of the digraph as

$$EE(D) = \sum_{k=0}^{\infty} \frac{\mathbf{D}^k}{k!} = \mathrm{tr}(e^{\mathbf{D}}) = \sum_{j=1}^{N} e^{\sigma_j}. \tag{10}$$

Using the Estrada indices of the digraph and its underlying graph the returnability for $c_k = 1/k!$ can be expressed as

$$\chi(c_k = 1/k!, G) = \frac{EE(D) - N}{EE[G(D)] - N}. \tag{11}$$

In the following result we show that the returnability, like the reciprocity, is bounded between zero and one.

Proposition 5. *The returnability of a digraph is bounded as* $0 \leqslant \chi(c_k) \leqslant 1$. *The lower bound is obtained for a graph containing no cycles and the upper bound is reached for symmetric digraphs.*

Proof. In general we can write the expression (2) defining the returnability of a digraph in terms of the eigenvalues of the adjacency matrices of the digraph and its underlying graph:

$$\chi(c_k, D) = \frac{c_2 \sum_{j=1}^{N} \sigma_j^2 + \cdots + c_3 \sum_{j=1}^{N} \sigma_j^3 + c_k \sum_{j=1}^{N} \sigma_j^k + \cdots}{c_2 \sum_{j=1}^{N} \lambda_j^2 + \cdots + c_3 \sum_{j=1}^{N} \lambda_j^3 + c_k \sum_{j=1}^{N} \lambda_j^k + \cdots}. \tag{12}$$

A digraph D contains no cycle if and only if the spectrum of D contains no eigenvalue different from zero, i.e., $\sigma_j = 0$ for $j = 1, \ldots, n$. Then, the numerator of (12) is zero, which proves the first part of the result. The second part of the result simply comes from the fact that a symmetric digraph is identical to its underlying graph. $\square$

It is interesting to investigate how the returnability is affected by the size of a directed cycle. For instance, in a symmetric digraph of two nodes the returnability is equal to one, but it decreases dramatically even for a directed triangle. Then, in general we have the following results for directed cycles of any length.

Proposition 6. *Let C_N be a directed cycle of order N. Then, the returnability* $\chi(c_k, C_N) \to 0$ *for* $N \to \infty$.

Proof. For directed cycles of order N the expression (2) can be written as follows:

$$\chi(c_k, C_N) = \frac{c_N N + c_{2N} N + \cdots + c_{kN} N + \cdots}{c_2 \mu_2[G(D)] + \cdots + c_{2k} \mu_{2k}[G(D)] + \cdots + c_N \mu_N[G(D)] + \cdots}. \tag{13}$$

Then, for large values of N, Eq. (13) can be approximated as

$$\chi(c_k, C_N) = \frac{N \sum_{k=1}^{\infty} c_{kN}}{N \sum_{k=1}^{\infty} c_{2k} \frac{(2k)!}{(k!)^2}}. \tag{14}$$

It can be easily shown that the denominator of (14) does not vanishes. Then, because the coefficients c_k are selected to decrease very fast as k increases, it is straightforward to realize that $\sum_{k=1}^{\infty} c_{kN} \to 0$ for $N \to \infty$, which proves the result. □

Remark 3. The denominator of (14) for the coefficients $c_k = 1/k!$ and $c_k = 1/N^k$ are approximated by $N \sum_{k=1}^{\infty} \frac{1}{(k!)^2} \approx 1.2795N$ and $N \sum_{k=1}^{\infty} \frac{(2k)!}{N^{2k}(k!)^2} \approx N$.

The last result has an important consequence for the study of returnability in complex directed networks. That is, a directed network displays large returnability if, and only if, it contains a large number of small directed cycles. These findings are analysed in the following section.

4. Numerical results

4.1. Returnability in small directed graphs

Small directed graphs are the basis of the structural motifs present in complex networks [34,35]. Network motifs are recurring, significant patterns of interactions, which appear more frequently in real-world complex networks than in random graphs with the same topology. The most studied motifs are three- and four-node directed graphs. We study here all 13 types of three-node connected directed graphs, which form the basis of the most important motifs. In Table 4.1, we give the values of the reciprocity and returnabilities of these 13 directed graphs.

As can be seen in Table 4.1, there are four non-returnable directed graphs, i.e., digraphs with zero returnability, with three nodes. Two of them, Nos. 2 and 4, have been identified as motifs in food webs and gene regulation networks, respectively. These graphs also have zero reciprocity. They receive the name of *three-chain* and *feed-forward loop*, respectively [34,35] (see Fig. 1). In general, the two measures of returnability as well as the reciprocity are correlated to each other for these small cycles. However, there are important differences in the one observed for the *three-node feedback loop* (see Fig. 1). This subgraph correspond to the number 5 in Table 4.1 and has been found as a motif in digital functional multipliers. It has no reciprocity but returnability different from zero. It is evident that non-returnability implies non-reciprocity but the reverse is false.

The graph 6 in Table 1 is known as the *uplinked mutual dyad* motif [34,35] and has been found to be significant in WWW. In the WWW network, the subgraphs 12 and 13 in Table 1, which display large returnability have been found to be among the three most significant motifs [34,35]. The graph 12 is named *feedback with two mutual dyads* and appears more than 100,000 times in the WWW version of

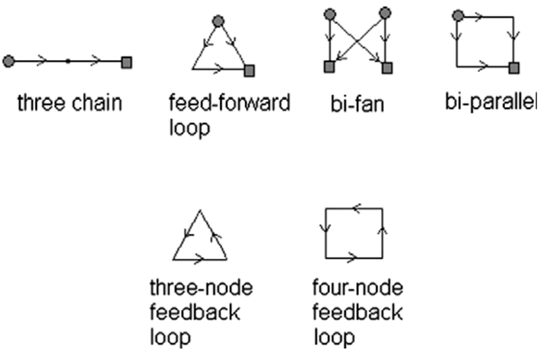
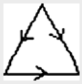
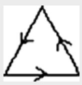
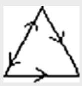


Fig. 1. Some subgraphs of three and four nodes that have been previously identified in the literature as network motifs.

Table 1
Values of the reciprocity and the returnability for the 13 directed graphs with 3 nodes.

Number	Subgraph	$EE(D)$	r	$\chi(c_k, G)$[a]	$\chi(c_k, G)$[b]
1		3.000	0.000	0.000	0.000
2		3.000	0.000	0.000	0.000
3		3.000	0.000	0.000	0.000
4		3.000	0.000	0.000	0.000
5		3.504	0.000	0.098	0.077
6		4.086	0.333	0.212	0.167
7		4.086	0.333	0.212	0.167
8		4.634	0.333	0.319	0.261
9		4.086	0.500	0.461	0.437
10		4.086	0.500	0.461	0.437
11		5.950	0.667	0.576	0.500
12		5.356	1.000	1.000	1.000
13		8.125	1.000	1.000	1.000

[a] $c_k = 1/k!$.
[b] $c_k = N^{-k}$.

nd.edu [34,35]. The graph 13, which is known as the *fully connected triad*, is the most populated motif in the WWW, where it appears almost 7 million times [34,35].

The motifs found by Milo et al. [34,35] for the communication network of WWW correspond to the ones having the largest returnability of all three-node graphs. This is due to the principal functional characteristic of this network which has been designed with the aim of having short paths between related pages in a returnable way to allow inter-users communicability. Thus, the use of the returnability parameter defined here can add some value to the analysis of the network motifs in complex networks, giving a quantitative support for the functionality of such significant patterns of interactions in complex directed networks.

4.2. Returnability in real-world directed networks

As the first examples for the analysis of returnability in real-world complex directed networks, we have selected two citation networks. In a citation network, the nodes represent the papers and the directed links the citations. That is, there is a link from paper A to B if the paper A cites the paper B. The first network is a citation network of papers published in the field of network centrality [36,37]. The second is a citation network of papers that cite S. Milgram's paper published in Psychology Today in 1967 or that use Small World in title [36,37]. In a citation network, there is no returnability for any node. That is, the citation networks are non-returnable graphs. If a paper A cites a paper B, it is because the paper B existed before paper A was created. Then, if B cites C, it is impossible that C cites A because it violates the law of causality. This effect prevents the presence of directed cycles in the graph, which makes it non-returnable. Then, as expected, both networks have returnability and reciprocity equal to zero.

Another example of non-returnable network is the Internet at the autonomous system (AS) level. An AS is a portion of the Internet under a single administrative authority. They can be administrated by a single institution, such as a company, a university, or an Internet service provider (ISP). We studied two versions of the Internet at 1997/11/08 and 1998/04/02 at the AS level [38]. In both cases, the directed networks are non-returnable with returnability and reciprocity equal to zero. These results reflect the real-world situation of the Internet, where the commercial relations between ASes make that the customer-provider relationships cannot contain (returnable) cycles [14,15]. For instance, a nation-wide AS may be a provider for a regional AS, which at the same time is a provider for a particular university campus, but this university campus cannot be a provider for the nation-wide AS. Note that a closed walk can exist in the underlying graph; consider e.g., the case that AS1 provides services to AS2 and AS3 and AS2 also provides services to AS3.

Then, we study other 12 real-world complex directed networks. They correspond to: (neurons) the neuronal synaptic network of the nematode *Caenorhabditis elegans*; (trans-yeast) the direct transcriptional regulation between genes in *Saccaromyces cerevisae*; four food webs corresponding to (St. Martin) birds and predators and arthropod prey of *Anolis* lizards on the island of St. Martin, which is located in the northern Lesser Antilles; (Benguela) a marine ecosystem of Benguela off the southwest coast of South Africa; (Skipwith) invertebrates in an English pond; (LittleRock) pelagic and benthic species, particularly fishes, zooplankton, macroinvertebrates, and algae of the Little Rock Lake, Wisconsin, US; (Roget) the vocabulary network of words related by their definitions in Roget's Thesaurus of English; (Prison) a social network of inmates in a prison; (electronics 1, 2 and 3) electronic sequential logic circuits, where nodes represent logic gates and flip-flops (digital fractional multipliers); (USAir97) the airport transportation network in the USA in 1997. The values of the reciprocity and returnability for these complex networks are given in Table 4.2.

As can be seen in Table 4.2 the values of the returnability for $c_k = N^{-k}$ are very close to those of the reciprocity. This is a consequence of the very large penalization imposed by the weighting scheme for large k which makes that practically only the closed walks of length 2 are considered in the returnability. However, the returnability based on $c_k = 1/k!$ is significantly different from the reciprocity in both qualitative and quantitative terms. This differences can be easily observed for the case of the three electronic circuits which have several cycles of length three but no cycles of length two. Then, these networks have reciprocity equal to zero but display certain returnability. In these electronic circuits, Milo et al. [34,35] found that the most important motifs present in such systems

Table 2
Values of the reciprocity and returnability measures for several complex directed real-world networks.

Network	Nodes	Links	r	$\chi(c_k, G)^a$	$\chi(c_k, G)^b$
Neurons	280	1973	0.0998	$7.795 \cdot 10^{-7}$	0.0994
Trans-yeast	662	1062	$9.416 \cdot 10^{-4}$	$4.637 \cdot 10^{-5}$	$9.434 \cdot 10^{-4}$
St. Martin	44	218	0.000	0.000	0.000
Benguela	29	191	0.0262	$1.966 \cdot 10^{-6}$	0.0165
Skipwith	35	353	0.0453	$3.756 \cdot 10^{-8}$	0.0239
LittleRock	181	2318	0.0206	$2.173 \cdot 10^{-15}$	0.0191
Prison	67	142	0.2820	0.1242	0.2804
Roget	994	3640	0.3895	0.0353	0.3894
Electronic1	122	189	0.0000	0.0164	$6.530 \cdot 10^{-4}$
Electronic2	252	399	0.0000	0.0153	$2.990 \cdot 10^{-4}$
Electronic3	512	819	0.0000	0.0145	$1.432 \cdot 10^{-4}$
USAir97	332	2126	1.0000	1.0000	1.000

[a] $c_k = 1/k!$.
[b] $c_k = 1/N^{-k}$.

correspond to the three-node feedback loop, bi-fan and the four-node feedback loop (see Fig. 1). The first and third motifs are returnable ones but they have relatively low values of returnability as we have already shown in the previous section for the three-node feedback loop. The bi-fan is a non-returnable motif, which explains the relatively low returnability of these circuits. The food webs studied display low returnability in full agreement with the results obtained by Milo et al., who have found that, in general, food webs contain large numbers of bi-parallel motifs as well as three chains (see Fig. 1), all of which are non-returnable motifs.

The three measures coincide in identifying the USAir97 network as fully-returnable and reciprocal, i.e., it is a symmetric digraph. This is a transportation network in which the returnability is essential for its correct function. However, the reciprocity and returnability based on $c_k = 1/k!$ differ in identifying the secondly ranked network. While the reciprocity identifies the Roget thesaurus network as the second most reciprocal, the returnability based on $c_k = 1/k!$ identifies the social network of prison as the second most returnable. In the thesaurus two words are connected if one is used in the definition of the other, which increases the probability of finding reciprocal relationships. In fact, 39% of the 3640 links are bi-directional [39]. In the social network this percentage is only 28% but in contrast it displays a large proportion of directed triangles and squares. Directed triangles are a consequence of the transitivity of social relationships and they have been established as a characteristic mark of social networks.

Finally, we can see in Table 4.1 that the biological and ecological networks display low reciprocity and returnability. The neuronal network has been found to have 125 feed-forward loops, 127 bi-fans and 227 bi-parallel motifs in this network (see Fig. 1) [34,35]. Neither of these structures is returnable, but consists of some firing neurons (circles in Fig. 1) and some sinks (squares in Fig. 1). It has been recognized that only 10% of synaptic couplings in this network are bi-directional. The yeast transcription network was found to have 70 feed-forward loops and 1812 bi-fan motifs, while the one of *Escherichia coli* has 40 and 203, respectively [34,35]. Here the existence of symmetric relationships and returnable cycles is very low, which makes practically no influence of the total returnability of these networks. Finally, Milo et al. [34,35] have found that food webs contain large numbers of bi-parallel motifs as well as three chains (see Fig. 1), all of which are non-returnable motifs.

5. Conclusion

The discovery of the unexpectedly coherent graphs representing disparate complex systems has placed “graph theory to the heart of a new paradigm of science” [40] in the XXI century. These complex graphs and digraphs (complex networks) are ubiquitous in society, biology, ecology, economy and modern technology. The understanding of their structure and functioning is vital to comprehend complex systems as a whole. Here we have introduced new measures that characterize an important architectural property of complex directed networks.

The concept of *returnability* introduced here is a generalization of the more limited concept of *reciprocity*, which has proved to be of remarkable importance in understanding complex networks. We have introduced two measures of returnability which are given by closed mathematical formulas for their calculation. More importantly, these measures can be expressed in terms of the graph spectrum, which allows to connect this concept with the vast arsenal of graph spectral theory. We finally have analysed the new concept and measures in studying a dozen of real-world complex networks arising in different scenarios. We have seen that the new concept of returnability complements very well other concepts used for studying complex networks, such as that of the *network motifs*. While network motifs are studied on a computational and statistical way, the returnability can be studied from a mathematical perspective. Thus, we agree with Chung and Lu [40] that “Mathematicians and especially graph theorists have much to contribute to building the scientific foundation” of complex networks.

Acknowledgments

EE thanks the support by the IIS, University of Tokyo for a fellowship as Research Visitor and for warm hospitality during April–June, 2008. EE thanks Prof. P. Rowlinson for a careful revision of the manuscript.

References

[1] R. Albert, A.-L. Barabási, Statistical mechanics of complex networks,Rev. Modern Phys. 74 (1) (2002) 47–97.
[2] M.E.J. Newman, The structure and function of complex networks, SIAM Rev. 45 (2003) 167–256.
[3] M.E.J. Newman, S.H. Strogatz, D.J. Watts, Random graphs with arbitrary degree distributions and their applications, Phys. Rev. E 64 (2001) 026118.
[4] S.N. Dorogovtsev, J.J.F. Mendes, A.N. Samukhin, Giant strongly connected component of directed networks, Phys. Rev. E 64 (2001) 025101R.
[5] A.D. Sánchez, J.M. López, M.A. Rodríguez, Nonequilibrium phase transitions in directed small-world networks, Phys. Rev. Lett. 88 (2002) 048701.
[6] H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, A.-L. Barabási, The large-scale organization of metabolic networks, Nature 407 (2000) 651–654.
[7] H. Ma, A.-P. Zeng, Reconstruction of metabolic networks from genome data and analysis of their global structure for various organisms, Bioinformatics 19 (2003) 270–277.
[8] R. Tanaka, Scale-rich metabolic networks, Phys. Rev. Lett. 94 (2005) 168101.
[9] J.E. Cohen, F. Briand, C.M. Newman, Community Food Webs: Data and Theory, Springer, Berlin, 1990.
[10] M. Pascual, J.A. Dunne (eds), Ecological Networks: Linking Structure to Dynamics in Food Webs, Oxford University Press, Oxford, 2006.
[11] D.A. Stouffer, J. Camacho, W. Jiang, L.A.N. Amaral, Evidence for the existence of a robust pattern of prey selection in food webs, Proc. Roy. Soc. B 274 (2007) 1931–1940.
[12] S. Wasserman, K. Faust, Social Network Analysis. Methods and applications, Cambridge University Press, Cambridge, 1994.
[13] B. Dutta, M.O. Jackson, The stability and efficiency of directed communication networks, Rev. Econom. Des. 5 (2000) 251–272.
[14] R. Pastor-Satorras, A. Vespignani, Evolution and Structure of the Internet. A Statistical Physics Approach, Cambridge University Press, Cambridge, 2004.
[15] L. Gao, On inferring autonomous system relationships in the Internet, IEEE/ACM Trans. Networking 9 (2001) 733–745.
[16] D. Garlaschelli, M.I. Loffredo, Patterns of link reciprocity in directed networks, Phys. Rev. Lett. 93 (2004) 268701.
[17] M. Boguñá, M.A. Serrano, Generalized percolation in random directed networks, Phys. Rev. E 72 (2005) 016106.
[18] V. Zlatić, M. Božičević, H. Štefančić, M. Domazet, Wikipedias: collaborative web-based encyclopedias as complex networks, Phys. Rev. E 74 (2006) 016115.
[19] M.Á. Serrano, A. Maguitman, M. Boguñá, A. Vespignani, Decoding the structure of the WWW: a comparative analysis of Web crawls, ACM Trans. Web 1 (2007) 10.
[20] D. Garlaschelli, M.I. Loffredo, Multispecies grand-canonical models for networks with reciprocity, Phys. Rev. E 73 (2006) 015101.
[21] G. Zamora-López, V. Zlatić, C. Zhou, H. Štefančić, J. Kurths, Reciprocity of networks with degree correlations and arbitrary degree sequences, Phys. Rev. E 77 (2008) 016106.
[22] E. Estrada, J.A. Rodríguez-Velázquez, Subgraph centrality in complex networks, Phys. Rev. E 71 (2005) 056103.
[23] J.A. de la Peña, I. Gutman, J. Rada, Estimating the Estrada index, Linear Algebra Appl. 427 (2007) 70–76.

[24] I. Gutman, A. Graovac, Estrada index of cycles and paths, Chem. Phys. Lett. 436 (2007) 294–296.
[25] Y. Ginosar, I. Gutman, T. Mansour, M. Schork, Estrada index and Chebyshev polynomials, Chem. Phys. Lett. 454 (2008) 145–147.
[26] R. Carbó-Dorca, Smooth function topological structure descriptors based on graph-spectra, J. Math. Chem. 44 (2008) 373–378.
[27] E. Estrada, N. Hatano, Statistical-mechanical approach to subgraph centrality in complex networks, Chem. Phys. Lett. 439 (2007) 247–251.
[28] E. Estrada, J.A. Rodríguez-Velázquez, Spectral measures of bipartivity in complex networks, Phys. Rev. E 72 (2005) 046105.
[29] E. Estrada, Spectral scaling and good expansion properties in complex networks, Europhys. Lett. 73 (2006) 649–655.
[30] D. Cvetković, P. Rowlinson, S. Simić, Eigenspaces of Graphs, Cambridge University Press, Cambridge, 1997.
[31] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer-Verlag, Berlin, 2002.
[32] K.B. Janiszowski, Inversion of square matrices in processors with limited calculation abilities, Int. J. Appl. Math. Comput. Sci. 13 (2003) 199–204.
[33] E. Estrada, D.J. Higham, Network properties revealed through matrix functions, University of Strathclyde Mathematics Research Report, 2008, #17.
[34] R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, U. Alon, Network motifs: Simple building blocks of complex networks, Science 298 (2002) 824–827.
[35] R. Milo, S. Itzkovitz, N. Kashtan, R. Levitt, S. Shen-Orr, I. Ayzenshtat, M. Sheffer, U. Alon, Superfamilies of evolved and designed networks, Science 303 (2002) 1538–1542.
[36] N.P. Hummon, P. Doreian, L.C. Freeman, Analyzing the structure of the centrality literature between 1948 and 1979, Sci. Comm. 11 (1990) 459–480.
[37] W. de Nooy, A. Mrvar, V. Batagelj, Exploratory Social Network Analysis with Pajek, Cambridge University Press, Cambridge, 2004.
[38] M. Faloutsos, P. Faloutsos, C. Faloutsos, On power-law relationships of the Internet topology, ACM SIGCOMM Comp. Comm. Rev. 29 (1999) 251–262.
[39] A. de Jesus Holanda, I. Torres Pisa, O. Kinouchi, A. Souto Martinez, E.E. Seron Ruiz, Thesaurus as a complex network, Physica A 344 (2004) 530–536.
[40] F. Chung, L. Lu, Complex Graphs and Networks, American Mathematical Society, Providence, 2006.