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Brief paper

Design of highly synchronizable and robust networks☆

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ABSTRACT

In this paper, the design of highly synchronizable, sparse and robust dynamical networks is addressed. Better synchronizability means faster synchronization of the oscillators, sparsity means a low ratio of links per nodes and robustness refers to the resilience of a network to the random failures or intentional removal of some of the nodes/links. Golden spectral dynamical networks (graphs) are those for which the spectral spread (the difference between the largest and smallest eigenvalues of the adjacency matrix) is equal to the spectral gap (the difference between the two largest eigenvalues of the adjacency matrix) multiplied by the square of the golden ratio. These networks display the property of "small-worldness", are very homogeneous and have large isoperimetric (expansion) constant, together with a very high synchronizability and robustness to failures of individual oscillators. In particular, the regular bipartite dynamical networks, reported here by the first time, have the best possible expansion and consequently are the most robust ones against node/link failures or intentional attacks.

1. Introduction

Complex networks are ubiquitous in our everyday life (Newman, 2003). They are formed by a large set of interconnected nodes representing the entities of the system (Wang & Chen, 2003). In particular, complex networked control systems represent an area of tremendous theoretical and practical interest (Abdallah & Tanner, 2007). In these systems the synchronization of all dynamical entities (nodes) is an interesting phenomenon with multiple practical applications, such as in the synchronous transfer of signals in communication networks (Chen & Zhou, 2006; Pavel, 2004; Ren, 2008; Su, Wang & Lin, 2009; Wu & Chua, 1995; Zhou, Lu, & Lü, 2006). Nowadays, it is well known that the structure of the network, e.g., 'small-worldness' (Watts & Strogatz, 1998) and 'scale-freeness' (Barabási & Albert, 1999), predetermines a large part of the dynamical processes taking place on it (Barahona & Pecora, 2002; Comellas & Gago, 2007; Lü, Yu, Chen & Cheng, 2004; Wang & Chen, 2002).

When studying synchronization of complex networks, it is usual to consider n identical oscillators represented by the nodes which are interconnected pairwise by means of the active links. Those links are considered to be equal. All communication is considered bidirectional, in such a way that the underlying network is undirected. Then, the time evolution of the ith oscillator is given by

$$\dot{x}_i = F(x_i) - \sigma \sum_k L_{ij} H(x_j), \tag{1}$$

where $x_{i=1,2,\ldots,n}$ is the state of the ith oscillator, F and H are the evolution and the output functions, respectively, σ is a coupling constant and L_{ij} are the entries of the discrete Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is the diagonal matrix of degrees and $\mathbf{A}$ is the adjacency matrix (Barahona & Pecora, 2002). In this case it has been shown that a network exhibits *good synchronizability* if the ratio $Q = \mu_1/\mu_{n-1}$ is as small as possible, where μ_1 and μ_{n-1} are the largest and the second smallest eigenvalue of the Laplacian matrix, respectively (Barahona & Pecora, 2002). The eigenvalue μ_{n-1} is frequently referred to as the algebraic connectivity of the graph (Fiedler, 1973).

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It has been shown that both scale-free and small-world networks display better synchronizability than regular graphs (Barahona & Pecora, 2002; Lü et al., 2004; Wang & Chen, 2002). However, it has been observed that networks with strong heterogeneity in the degree distribution are much more difficult to synchronize than random homogeneous networks (Nishikawa, Motter, Lai, & Hoppensteadt, 2003). Consequently, the necessity for designing highly robust networks having the ability of easily synchronize individual processes taking place at their nodes is an urgent necessity in this field (Donetti, Hurtado & Muñoz, 2006; Motter, Zhou, & Kurths, 2005).

One strategy that has been intensively explored in system control theory is the use of pinning control. This strategy is useful when the whole network cannot synchronize by itself. Then, some controllers can be used to force the network to synchronize. The pinning control strategy consists in applying some local feedback injections to a fraction of the nodes (Chen, Chen, Xiang, Liu, & Yuan, 2009; Porfiri & di Bernardo, 2008; Yu, Chen, & Lü, 2009; Zhou, Lu, & Lü, 2008). A different approach was proposed by Donetti et al. (2006) based on the use of entangled networks. They obtained by a numerical optimization algorithm, some robust and highly synchronizable networks. These networks are extremely homogeneous, with long cycles and poor modular structure. Using numerical optimization algorithms Donetti et al. (2006) have found some networks which are highly synchronizable and robust. Here we propose a radically different approach which is based on the so-called *golden spectral graphs* (GSGs) (Estrada, 2007). These graphs can be built using analytical tools instead of numerical methods, which allows the construction of infinite series of such graphs. These graphs are also super-homogeneous, display "small-world" properties, good expansion properties in the graph-theoretic sense and more importantly they have high synchronizability. The construction of these networks in the current work is carried out by means of matrix operations. Here we show some of the topological properties of these networks including their synchronizability and robustness to nodes/links random failures and intentional attacks.

2. Preliminary definitions

Let $G = (V, E)$ be a simple connected graph of order $n = |V|$. As usual $\mathbf{A}$ stands for the adjacency matrix of the graph and its associated spectrum is denoted by

$$sp(\mathbf{A}) = \{\lambda_1^{m_1} > \lambda_2^{m_2} > \cdots > \lambda_d^{m_d}\}, \qquad n = \sum_{i=1}^{d} m_i,$$

where λ_i is the ith eigenvalue with m_i multiplicity. Let $s(G) = \lambda_1 - \lambda_d$ be the *spread*, $g(G) = \lambda_1 - \lambda_2$ the *spectral gap* and $w(G) = \lambda_2 - \lambda_d$ the *width* of the "bulk" part of the spectrum of G. In the case of regular networks, i.e., those having the same degree for every node, the Laplacian eigenvalues are related to the ones of the adjacency matrix as $\mu_j = \lambda_1 - \lambda_{n-j+1}$

The expansion constant or isoperimetric number (Mohar, 1989) is a measure of how efficiently connected a network is and it is defined as follows

$$\phi(G) = \inf\left\{\frac{|\partial S|}{|S|}, S \subseteq V, 0 < |S| \leq \frac{|V|}{2} < +\infty\right\}, \tag{2}$$

where $|\partial S|$ denotes the boundary of S, which is the number of links that connect a node in S with a node in $V - S$. The subset S is selected to be at most half the number of nodes in the network. For good expansion networks (GENs) this constant should always be larger than a given positive number ε. It is known that a network has GE if the gap between the largest λ_1 and second largest λ_2 eigenvalues of the adjacency matrix $g(\mathbf{A}) = \lambda_1 - \lambda_2$ is sufficiently large. The expansion constant and the spectral gap are related by the well-known Alon–Milman inequality (Alon & Milman, 1985),

$$\frac{g(\mathbf{A})}{2} \leq \phi(G) \leq \sqrt{2\lambda_1 g(\mathbf{A})}. \tag{3}$$

Thus, the larger the spectral gap, the larger the expansion constant of the graph. Among the graphs with larger spectral gap there is a family of graphs named *Ramanujan graphs* (Lubotzky, Phillips, & Sarnak, 1988). They are defined as the d-regular graphs for which $\lambda_2 \leq 2\sqrt{d-1}$, where λ_2 is the maximum of the non-trivial eigenvalues of the graph $\lambda(G) = \max_{|\lambda_i|<d}|\lambda_i|$. These graphs, which are the best possible expanders, have found applications in network design, complexity theory, derandomization, coding theory and cryptography. The interested reader is referred to the excellent review (Hoory, Linial, & Wigderson, 2006).

It is well known that in the case of random networks the largest eigenvalue grows much faster than the second largest one: $\lim_{n\to\infty}(\lambda_1/n) = p$ with probability 1, while for any $\varepsilon > 1/2$, $\lim_{n\to\infty}(\lambda_2/n^{\varepsilon}) = 0$. A similar relation holds for the smallest eigenvalue as well. This means that the spectral gap grows very fast while the bulk of the spectrum is concentrated in a semi-circle demarked by $w(\mathbf{A}) = \lambda_2 - \lambda_d$. Similar situations have been observed for "small-world" graphs as well as for "scale-free" networks (Farkas, Derényi, Barabási, & Vicsek, 2005). In the case of the bulk part of the scale-free graphs the spectral density is triangle like instead of semi-circular.

3. Motivations

The main motivation for the current work is the seminal result of Barahona and Pecora (2002) showing that a network exhibits good synchronizability if the ratio $Q = \mu_1/\mu_{n-1}$ is as small as possible. We are interested here only in regular networks. Consequently, the spectral ratio can be expressed in terms of the eigenvalues of the adjacency matrix of the network as

$$Q = \frac{\lambda_1 - \lambda_d}{\lambda_1 - \lambda_2}, \tag{4}$$

where the numerator is the spectral spread $s(G)$ and the denominator the spectral gap $g(G)$ of the network. Then, in order to design highly synchronizable networks we need to reduce Q by decreasing the spectral spread and increasing the spectral gap. A simultaneous change of both parameters, e.g., increasing the spectral spread and decreasing the gap, produces a shrink of the spectral width $w(G) = \lambda_2 - \lambda_d$. Consequently, we would like to represent the spectral ratio Q in term of the three main spectral parameters we have defined in the previous section: spectral spread, gap and width. This can be easily done if we consider networks which are neither complete nor empty. In a complete network every pair of nodes are connected and $w(G) = 0$. In the empty network there are no links and $g(G) = 0$. For any other network $w(G) \neq 0$ and $g(G) \neq 0$ and we can multiply and divide Q by the spectral width to obtain

$$Q = \left(\frac{\lambda_2 - \lambda_d}{\lambda_1 - \lambda_2}\right)\left(\frac{\lambda_1 - \lambda_d}{\lambda_2 - \lambda_d}\right), \tag{5}$$

where we can define the two terms in parentheses as two spectral ratios

$$\omega_1(G) = \frac{\lambda_2 - \lambda_d}{\lambda_1 - \lambda_2}, \tag{6}$$

$$\omega_2(G) = \frac{\lambda_1 - \lambda_d}{\lambda_2 - \lambda_d}. \tag{7}$$

We note in passing that the quantity $R = 1/\omega_1 = (\lambda_1 - \lambda_2)/(\lambda_2 - \lambda_n)$ was proposed as a measure of the distance of the first

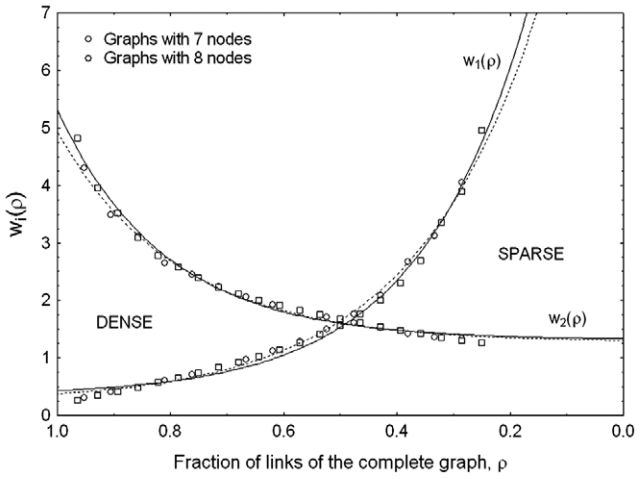

Fig. 1. Spectral ratios as function of the density for all connected graphs with 7 and 8 nodes.

eigenvalue from the main part of the distribution of eigenvalues normalized by the extension of the main part (Farkas et al., 2005).

A simple analysis of Eq. (5) reveals that changing the spectral width as a way to decrease Q and increase synchronizability is not trivial. For instance, decreasing $w(G)$ decreases $\omega_1(G)$, which also decreases Q. However, it increases $\omega_2(G)$ at the same time, which has the contrary effect on the values of Q and synchronizability.

Another factor that should be taken into account in designing highly synchronizable network is their density of links per nodes. For instance, a complete network has the best possible synchronizability with $Q = 1$. However, in practical terms it represents a high cost to connect every pair of nodes to each other. Consequently, we are interested in finding highly synchronizable networks which are also sparse. We use a definition of density previously studied by Barahona and Pecora (2002) as

$$\rho(G) = \frac{2|E|}{n(n-1)}, \tag{8}$$

which represents the fraction of links a network has ($|E|$) respect to the number of links in a complete network with the same number of nodes.

The problem of designing a highly synchronizable network under the current scheme can now be stated as follows:

Problem. Find a network for which $\omega_1(G)$ and $\omega_2(G)$ are as small as possible by keeping the density of the network low.

4. Basic principles for design

In order to gain some insights about the parameters we need to use for our design of highly synchronizable sparse networks we investigate the following. We selected all 853 connected networks having 7 nodes and all 11,117 connected networks having 8 nodes. For both groups of networks we obtain the average values of the spectral ratios for networks with the same densities: $\omega_1(\rho)$ and $\omega_2(\rho)$, and plot them versus density in Fig. 1.

It is straightforward to realize that $Q \equiv \omega_1 + 1$. Then, it can be seen in Fig. 1 that the best synchronizability is obtained for dense graphs, e.g., $\rho > 0.5$ (left part of the figure). On the contrary, sparse networks ($\rho < 0.5$) display very high values of Q, which means poor synchronizability.

The best possible scenario, i.e., high synchronizability and sparsity, is obtained when $\omega_1(G) = \omega_2(G)$. It is easy to show that this condition is fulfilled if, and only if:

$$\omega_1(G) = \omega_2(G) = \varphi,$$

where $\varphi = (1+\sqrt{5})/2 \approx 1.60803399$ is the golden ratio.

Consequently, the networks with best synchronizability which display density far from the complete graph are golden spectral graphs (GSGs), which are those defined as follows:

Definition 1 (*Estrada, 2007*)**.** A golden spectral network is a graph for which both spectral ratios are identical, that is

$$\omega_1(G) = \omega_2(G) = \varphi.$$

Then, it is obvious that for a d-regular graph which is GSG: $Q(GSG) = \varphi^2 = \varphi + 1 \approx 2.6180\ldots$. We recall that from the networks found by Donetti et al. (2006) only one, the Petersen graph, displays $Q < \varphi^2$.

Now, the fundamental result we obtained here is a method that allows us to build new GSGs from known ones. That is, this method guaranties that once we have identified a GSG we can build infinite series of GSGs, all of them displaying good synchronizability and sparsity. This result is stated below, in which we use some basic algebraic operations that allow to extend a GSG to a family of similar graphs. Then, first we define such operations as follows:

(1) The tensor product of two graphs $G \otimes G'$ is defined as the tensor (Kronecker) product of the adjacency matrices of both graphs,
(2) The graph $G \circledast J_k$ denotes the graph with adjacency matrix $(\mathbf{A}+\mathbf{I}) \otimes \mathbf{J}_k - \mathbf{I}$, where $\mathbf{J}_k$ is the all-ones matrix of order k.

Theorem 1. *Let G be a GSG. Then, for any $r \geq 1$*

(1) *$G \otimes J_r$ is GSG,*
(2) *$G \circledast J_r$ is GSG.*

Proof. Recall that $sp(\mathbf{J}_r) = \{0^{r-1}, r^1\}$ and $sp(G \otimes J_r)$ is the product of the eigenvalues of both graphs (Cvetković, Doob, & Sach, 1982). Also observe that $0 > \lambda_d$. Then, the non-zero eigenvalues verify

$$r\lambda_1^{m_1} > \cdots > r\lambda_d^{m_d},$$

and therefore it is easy to check that

$$\omega_1(G \otimes J_r) = \varphi.$$

On the other hand, the non-zero eigenvalues of $G \circledast J_r$ satisfy $(r\lambda_1 + r - 1)^{m_1} > \cdots > (r\lambda_d + r - 1)^{m_d}$ (Van Dam, 1995). Consequently, $\omega_1(G \circledast J_r) = \varphi$.

For the sake of completeness it is necessary to remark that for any graph it holds that

$$\omega_1(G) = \varphi \Leftrightarrow \omega_2(G) = \varphi.$$

To prove it we only need to suppose that $\omega_1(G) = \varphi$, then $\lambda_n = -\varphi\lambda_1 + \varphi^2\lambda_2$. Consequently,

$$\omega_2(G) = \frac{\lambda_1 + \varphi\lambda_1 - \varphi^2\lambda_2}{\lambda_2 + \varphi\lambda_1 - \varphi^2\lambda_2} = \frac{(1+\varphi)}{\varphi}\frac{S(\mathbf{A})}{S(\mathbf{A})} = \varphi. \quad \square$$

The use of tensor product for generating new from old graphs is justified by the fact that the spectra of the new and old graphs are related in such a way that does not affect the properties of being GSG as shown in the proof of this theorem. In addition, this procedure has been proposed recently as a standard way of generating growing networks with desired properties (Leskovec, Chakrabarti, Kleinberg, Faloutsos, & Ghahramani, 2009).

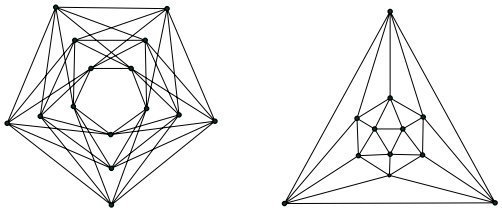

Fig. 2. Two examples of GSGs: $C_5 \otimes J_3$ and the icosahedral graph.

Table 1
The different types of networks built using Theorem 1 together with their sizes n and densities ρ.

Network	n	$\rho = d/(n-1)$
$(C_5 \otimes J_k) \circledast J_r$	$5kr$	$\frac{2rk+r+1}{5kr-1}$
$(C_5 \circledast J_k) \otimes J_r$	$5kr$	$\frac{3kr-r}{5kr-1}$
$(C_3 \circledast J_k) \circledast J_r$	$3k^2r$	$\frac{3kr-1}{3k^2r-1}$
$(C_3 \circledast J_k) \otimes J_r$	$3k^2r$	$\frac{3kr-r}{3k^2r-1}$
$(C_5 \circledast J_k) \circledast J_r$	$5k^2r$	$\frac{3kr-1}{5k^2r-1}$
$(C_5 \circledast J_k) \otimes J_r$	$5k^2r$	$\frac{3kr-r}{5k^2r-1}$

5. New GSGs from old ones

It is clear from Theorem 1 that once we find a GSG we can build infinite series of such networks simply by applying some algebraic operations to their adjacency matrices. The smallest possible GSG was previously found by Estrada (2007), which is the 2-regular graph with 5 nodes, i.e., a pentagon. We have also previously shown (Estrada, 2007) that the graphs $C_5 \otimes J_k$ are GSG for any $k \geq 1$, and that the k-covers of $C_3 \circledast J_k$ and $C_5 \circledast J_k$ are GSG. Here we represent the k-covers of $C_3 \circledast J_k$ and $C_5 \circledast J_k$ as $\overline{C_3 \circledast J_k}$ and $\overline{C_5 \circledast J_k}$, respectively. These graphs have the following adjacency matrices (Van Dam, 1995)

$$\mathbf{A}_3 = \begin{pmatrix} \mathbf{D} & \mathbf{P} & \mathbf{P}^T \\ \mathbf{P}^T & \mathbf{D} & \mathbf{P} \\ \mathbf{P} & \mathbf{P}^T & \mathbf{D} \end{pmatrix},$$

$$\mathbf{A}_5 = \begin{pmatrix} \mathbf{D} & \mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{P}^T \\ \mathbf{P}^T & \mathbf{D} & \mathbf{P} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^T & \mathbf{D} & \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}^T & \mathbf{D} & \mathbf{P} \\ \mathbf{P} & \mathbf{0} & \mathbf{0} & \mathbf{P}^T & \mathbf{D} \end{pmatrix},$$

where $\mathbf{P}$ be the $k^2 \times k^2$ matrix defined as follow (Van Dam, 1995)

$$\mathbf{P} = \begin{pmatrix} \mathbf{I} & \mathbf{I} & \cdots & \mathbf{I} \\ \mathbf{C} & \mathbf{C} & \cdots & \mathbf{C} \\ \vdots & \vdots & & \vdots \\ \mathbf{C}^{k-1} & \mathbf{C}^{k-1} & \cdots & \mathbf{C}^{k-1} \end{pmatrix},$$

and $\mathbf{C}$ is the $k \times k$ circulant matrix whose elements are $C_{ij} = 1$ if $j = i + 1 \pmod k$, and $C_{ij} = 0$ otherwise. On the other hand, $\mathbf{D} = (\mathbf{J}_k - \mathbf{I}_k) \otimes \mathbf{I}_k$. A particular case of these covers is the icosahedral graph, which is $\overline{C_3 \circledast J_2}$. Two examples of golden spectral graphs constructed by the previous methods are illustrated in Fig. 2.

A generalization of these results comes now from Theorem 1, which proves that the graphs $C_5 \circledast J_k$ are GSGs because we have previously proved that the pentagon C_5 is GSG (Estrada, 2007). Now using Theorem 1, the previous finding, and the construction methods developed here we can build several new families of GSGs. Excluding the trivial case $(C_5 \otimes J_k) \otimes J_r = C_5 \otimes J_{kr}$, we have six new series of networks. They are the d-regular graphs of size n illustrated in Table 1.

Obviously, all the graphs displayed in Table 1 are among the best synchronizable networks that have been reported in the literature so far. For instance, the best synchronizers obtained by Donetti et al. (2006) are cage graphs known as *Petersen*, *Heawood* and *McGee* graphs. These graphs have Q ratios equal to 2.500, 2.784, and 5.562, respectively. This means that only the Petersen graph displays better synchronizability than the GSGs. On the other hand, the densities of these networks tend to very low values as the number of nodes tends to infinite. For instance, the graphs given in the first two entries of Table 1 have densities that tends to 0.4 and 0.6, respectively when $n \to \infty$. However, the rest of GSGs in this table have densities that tends to zero as $n \to \infty$, which means that they are very sparse indeed. In closing, networks given in Table 1 are very sparse and among the highest synchronizable reported in the literature so far.

6. Properties of GSGs

An important property that networks should display for practical purposes is the one of robustness. Robustness to random failures and intentional attacks can be related to the property of good expansion, which is connected to the spectral gap by the Alon–Milman theorem given in Section 2. The best possible expansion is obtained for the so-called Ramanujan graphs (see Section 2). It is known that the Petersen graph, which displays an excellent synchronizability, is Ramanujan. However, other graphs found as good synchronizers like *Heawood* or *McGee* graphs are not Ramanujan. In the following we are going to analyze the properties related to the expansibility or isoperimetric parameter for the GSGs.

Here we improve the bounds for the expansion constant given by the Alon–Milman theorem (Alon & Milman, 1985) for the case of GSGs. This result can be obtained from the bound found by Mohar (1989) for the isoperimetric number in terms of the algebraic connectivity. Mohar (1989) found that $\phi(G) > \mu_{n-1}/2$. Then, by combining this result with the one showing that μ_1 is bounded by the maximum degree Δ of the nodes in the graph as $\Delta < \mu_1 < 2\Delta$ (Fiedler, 1973), we can obtain a lower bound for the isoperimetric number of d-regular GSGs. On the other hand, from the Cheeger inequality, which relates the isoperimetric number of a graph with its algebraic connectivity and its maximum degree (see Mohar, 1989) we have that

$$\mu_{n-1}^2 - 2\Delta\mu_{n-1} + \phi(G)^2 \leq 0,$$

which leads us to

$$\frac{1}{\Delta + \sqrt{\Delta^2 - \phi(G)^2}} \leq \frac{1}{\mu_{n-1}} \leq \frac{1}{\Delta - \sqrt{\Delta^2 - \phi(G)^2}}.$$

Combining the upper bound with $\Delta < \mu_1 < 2\Delta$ we obtain an upper bound for the isoperimetric number of a d-regular GSG. Both, the lower and upper bounds previously referred are given in the following result.

Theorem 2. *Let G be a d-regular GSG. The expansion or isoperimetric constant is bounded as follows*

$$\frac{d}{2\varphi^2} \leq \phi(G) \leq \frac{2d\sqrt{\varphi}}{\varphi^2}.$$

This result shows that the d-regular GSG have large expansion. This means that such networks are quite robust to random failures as well as to the intentional removal of nodes and links. To put this in context we have to say that a network with good expansion lacks topological bottlenecks, which are sets of nodes/links whose removal separate the network into isolated parts. The higher the expansion coefficient, the larger the robustness of the graph against node/link removals. Among the graphs found there are several Ramanujan graphs, i.e., $\lambda_2 \leq 2\sqrt{d-1}$, which are the

graphs having the largest possible expansion. Concerning the intentional removal of nodes and links we have to add that GSGs are super-homogeneous, i.e., all nodes look similar in their topological properties, which makes difficult if not impossible to select some of them as possible targets for an intentional attack. For the sake of illustration we consider the graph $(C_3 \circledast J_2) \otimes J_5$, which is regular of degree 25. In this graph all nodes have the same betweenness (19.5), closeness centrality (60.204) and clustering coefficient (0.417). In addition, all links have the same betweenness centrality (3.92). For the definition of the centrality measures the reader is referred to Wasserman and Faust (1994).

The third group of the topological properties characterized here for GSGs is related to distance-based parameters. The most important distance-based parameters that we study here are the average shortest path length $\bar{l}$ and the diameter D, which is the maximum distance separating any pair of nodes in the network. These results are obtained by using some bounds previously obtained by Mohar (1991). If we consider two subsets of nodes, B and C, separated at distance $r+1$, Mohar (1991) has shown that

$$Q > 4(r-1)^2 \frac{|B||C|}{(n-|B|-|C|)(|B|+|C|)},$$

where, as before, $Q = \mu_1/\mu_{n-1}$ and $|\cdots|$ stands for the cardinality of the subsets. If we consider that both subsets contain only one node each, which are separated at the maximum distance D, the previous bound turns out into

$$Q > \frac{2(D-2)^2}{(n-2)}.$$

Then, by considering a d-regular GSG we can obtain the following bound for the distance separating the two subsets of nodes,

$$(r-1)^2 < \frac{\varphi^2}{4} \frac{(n-|B|-|C|)(|B|+|C|)}{|B||C|},$$

from which we obtain the following result for the diameter of a d-regular GSG.

Theorem 3. *Let G be a d-regular GSG. Then, the diameter D is bounded as*

$$D < 2 + \frac{\sqrt{2}}{2}\varphi\sqrt{n-2.}$$

We further adapt the bounds found by Mohar (1991) relating Q with the diameter and the average shortest path distance to GSGs. The reader is referred to the work of Mohar (1991) to see the derivation of these bounds.

Theorem 4. *Let G be a d-regular GSG with n nodes. Then, the average shortest path length $\bar{l}$ and the diameter D are bounded as follows*

$$\bar{l} < \frac{n}{n-1}\left\lceil 1+\varphi\sqrt{\frac{\alpha^2-1}{4\alpha}}\right\rceil\left(\frac{1}{2}+\left\lceil \log_\alpha \frac{n}{2}\right\rceil\right),$$

$$D < 2\left\lceil \varphi\sqrt{\frac{\alpha^2-1}{4\alpha}}+1\right\rceil\left\lceil \log_\alpha \frac{n}{2}\right\rceil,$$

where $\alpha > 1$ is a parameter.

These two results show that d-regular GSGs are "small-world" graphs. In such graphs, the average path length scales as $\bar{l} \sim \log(n)$, which means that there is a relatively small separation between any pair of nodes in the network in comparison with the size of the graph. For instance, the average shortest path length of the graph $(C_3 \circledast J_2) \otimes J_5$ is only $\bar{l} = 1.661$, which is lower than $\log(n)$ for this network of 60 nodes.

Finally, we show a lower bound for the chromatic number of a GSG following a well-known result of Hoffman (1970). The chromatic number of a graph G is the smallest number of colors $\chi(G)$ needed to color the vertices of G so that no two adjacent vertices share the same color. Graph coloring have been applied to many different problems. For instance, the problems of scheduling and timetabling, computer register allocation, municipal waste collection, mobile radio frequency assignment, and computation of sparse Jacobian elements (For a review see Butenko, Festa, & Pardalos, 2001).

Theorem 5. *Let G be a GSG with n nodes and let $\chi(G)$ be the chromatic number. Then,*

$$\chi(G) \geq \varphi\left(1-\frac{\lambda_2}{\lambda_n}\right).$$

Proof. If G is a GSG then

$$\lambda_2-\lambda_n = \varphi(\lambda_1-\lambda_2) \Rightarrow \varphi\lambda_2+\lambda_2-\lambda_n = \varphi\lambda_1$$
$$\Rightarrow -\frac{\varphi^2\lambda_2}{\varphi\lambda_n}+\frac{\lambda_n}{\varphi\lambda_n} = -\frac{\varphi\lambda_1}{\varphi\lambda_n}$$
$$\Rightarrow \varphi\left(-\frac{\lambda_2}{\lambda_n}+1\right) = 1-\frac{\lambda_1}{\lambda_n} \leq \chi(G). \quad \square$$

7. Computational search of bipartite GSGs

After having found several classes of GSGs by using analytical techniques in the previous section we embarked now in a computational search of this type of graphs. In general, a graph can be recognized as a GSG due to the following property of its spectral gap and spread.

Lemma 6. *A graph G is GSG if, and only if, the spectral spread is related to the spectral gap as follow*

$$s(\mathbf{A}) = \varphi^2 g(\mathbf{A}).$$

Proof. Let us take $s(\mathbf{A}) = \alpha$ and $g(\mathbf{A}) = \beta$. Then, $\omega_1(G) = (\alpha-\beta)/\beta$ and $\omega_2(G) = \alpha/(\alpha-\beta)$. If G is a GSG then $\omega_1(G) = \omega_2(G) = \varphi$. We take $\alpha/(\alpha-\beta) = \varphi$, which can be expressed as $1-\alpha/\beta = \varphi$. Consequently, $\alpha/\beta = \varphi+1 = \varphi^2$, which means $s(\mathbf{A}) = \varphi^2 g(\mathbf{A})$. If $s(\mathbf{A}) = \varphi^2 g(\mathbf{A})$ then $\alpha = \varphi^2\beta$, which means that $\omega_1(G) = (\alpha-\beta)/\beta = (\varphi^2\beta-\beta)/\beta = \varphi$ and $\omega_2(G) = \alpha/(\alpha-\beta) = \varphi^2\beta/(\varphi^2\beta-\beta) = \varphi$. $\square$

In particular we are interested here in the search of regular bipartite graphs with GS properties. The main reason for this search is that we have previously proved that every bipartite d-regular GSG is Ramanujan for $d \leq 70$ (Estrada, 2007). In addition, up to now no one bipartite GSG has been reported whatsoever and no analytical tool exist for building them *ad hoc*.

We have developed a computer program that search for d-regular bipartite GSGs based on the following theoretical principle.

Lemma 7. *A bipartite graph is GSG if $g(\mathbf{A}) = 2\varphi^{-2}\lambda_1$.*

Proof. In a bipartite graph $\lambda_1 = -\lambda_n$. Then $\omega_2(G) = \frac{2\lambda_1}{\lambda_2+\lambda_1}$, which for the GSG means $\frac{2\lambda_1}{\lambda_2+\lambda_1} = \varphi$. This can be rewritten as $\lambda_2 = \frac{2-\varphi}{\varphi}\lambda_1$, from which the result follows straightforwardly.

From this search we automatically remove the complete bipartite graphs, for which $w_1(K_{a,b}) = 1$ and $w_2(K_{a,b}) = 2$.

The overall principle underlying the program developed here is the one developed for the system *AutoGraphiX* (for a review see Aouchiche et al., 2006). However, instead of searching for

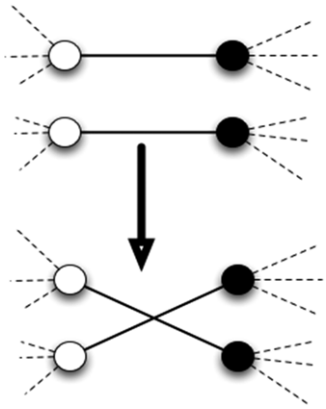

Fig. 3. Illustration of a two-opt transformation that preserves degrees and bipartiteness.

graphs and checking their properties afterward, we decided to take advantage of the highly constrained structure of the regular bipartite GSGs to reduce the search space and allows the program to properly work with more than 30 or 40 vertices. This size of graphs is traditionally difficult to handle with the regular version of AutoGraphiX.

Technically, the search for regular bipartite graphs with the appropriate spectral property specified by Lemma 7 (for a given number of vertices and degree) is carried out in two phases, which are explained below:

1. Find a regular bipartite graph with given degree. White and black vertices are fixed *a priori* and no edge joins two vertices of the same color. Some basic transformations such as add an edge, remove an edge, rotate an edge or a special combination of these are then considered to find a first bipartite regular graph with given degree.
2. Find a graph with the correct spectral property which is among bipartite regular graphs with given degree by minimizing the error function

$$err(G) = \lambda_1 - \lambda_2 - \lambda_1/\varphi^2.$$

To improve the efficiency of the search, the only transformation that is used at this step is a two-opt that preserves bipartiteness as illustrated in Fig. 3.

Using this transformation, if G is a bipartite regular graph, then any graph G' obtained from G will also be bipartite and regular (of the same degree). The connectivity is ensured by penalizing graphs that are not connected (adding a large value to the function $f(G)$). During the search, a graph is transformed if and only if $err(G)$ is reduced. The search is stopped either when a graph cannot be improved or when $err(G) = 0$ (in which case we found the correct graph). This search is heuristic and does not guarantee that a proper graph would systematically be found, even if it does exists. However, to improve the performance of the program, it is embedded within a Variable Neighborhood Search (VNS) metaheuristic (Hansen & Mladenovic, 2001; Mladenovic & Hansen, 1997) that applies increasing magnitude perturbations to the current best graph (preserving its structural properties) followed by local searches. The principle of the VNS used here is the same as in AutoGraphiX and a detailed description could be found in Caporossi and Hansen (2000).

Using this computer program we have found several d-regular bipartite GSGs. One example is a graph having two disjoint sets of 17 nodes all having degree 12, with eigenvalues $\lambda_1 = 12$, $\lambda_2 = 2.8329$ and $\lambda_n = -12$, diameter $D = 3$ and average shortest path length $\bar{l} = 1.788$. Another example is a regular bipartite graph having 34 nodes and degree 13 which was found in a similar way. This graph has $\lambda_1 = 13$, $\lambda_2 = 3.0689$ and $\lambda_n = -13$. The adjacency matrices of these graphs are given in the Appendix.

The advantage of finding bipartite GSGs by using a computational search is that we can build other bipartite GSGs from them by analytical results which use graph operations. In the following we give a pair of such results. The first is a corollary of the Theorem 1.

Corollary 8. *Let G be a bipartite GSG with adjacency matrix $\mathbf{B}$. Then, the graph obtained as $G \otimes J_k$ is also a bipartite GSG.*

We have built the graphs $G \otimes J_k$ where G is the graph with 34 nodes and degree 12 given in the Appendix. In this case all graphs constructed are Ramanujan for $k \leq 5$. All graphs have diameter $D = 3$ and average shortest path length $\bar{l} < \log(n)$. Consequently, they can be considered as "small-world" bipartite Ramanujan graphs with high synchronizability, recall that $Q \approx 2.618$ for any regular GSG. The same properties are reproduced for the graphs $G \otimes J_k$ where G is the regular bipartite GSG having 34 nodes and degree 13 mentioned before, which is also given in the Appendix.

The second result allows the generation of new regular bipartite GSGs by finding the product of two bipartite graphs.

Lemma 9. *Let G_1 be a regular bipartite GSG with spectra $sp(\mathbf{A}_1) = \{\lambda_1^{m_1} > \lambda_2^{m_2} > \cdots > \lambda_d^{m_d}\}$, $n_1 = \sum_{i=1}^{d} m_i$, and let G_2 be a regular bipartite graph with spectra $sp(\mathbf{A}_2) = \{\eta_1^{m'_1} > \eta_2^{m'_2} > \cdots > \eta_d^{m'_{d'}}\}$, $n_2 = \sum_{i=1}^{d'} m'_i$. If $\lambda_2\eta_1 < \lambda_1\eta_2$, then the graph $G_1 \times G_2$ is a graph with two bipartite GSGs components.*

Proof. Recall that the adjacency matrix of the product $G_1 \times G_2$ of two graphs is the Kronecker product of the adjacency matrices of G_1 and G_2. On the one hand, the product of two bipartite graphs gives a disconnected graph with two bipartite components (Theorem 9 in Leskovec et al. (2009)). On the other hand, the spectrum of the product of two graphs is the product of all their eigenvalues (Cvetković et al., 1982). Because G_1 and G_2 are bipartite graphs, $\lambda_1 = -\lambda_n$ and $\eta_1 = -\eta_n$. Then, assuming that $\lambda_2\eta_1 < \lambda_1\eta_2$, the spectrum of the product is $-\lambda_1\eta_1 \leq \cdots \leq \lambda_1\eta_2 \leq \lambda_1\eta_1$, and

$$\omega_1(G_1 \times G_2) = \frac{\lambda_1\eta_2 + \lambda_1\eta_1}{\lambda_1\eta_1 - \lambda_1\eta_2} = \frac{\eta_2 - \eta_n}{\eta_1 - \eta_2} = \varphi. \quad \square$$

For instance the product of the two regular bipartite GSGs given in the Appendix is formed by two identical regular bipartite graphs having 578 nodes, degree 156, diameter $D = 3$ and average shortest path length $\bar{l} = 1.9601$, with $\lambda_1 = 156$, $\lambda_2 = 36.8274$ and $\lambda_n = -156$. This graph is a "small-world" regular bipartite GSG but is not Ramanujan.

Another possibility of building new bipartite GSGs is by combining Corollary 8 and Lemma 9. In this case we can obtain regular bipartite GSGs by using $(G_1 \otimes J_k) \otimes (G_2 \otimes J_r)$. $\square$

8. Conclusions

In this paper, we have proposed several methods to building golden spectral graphs (GSGs). These graphs display high synchronizability of dynamical processes occurring at their nodes. In addition, GSGs are highly sparse networks displaying an excellent robustness against random failures as well as against intentional attacks of both nodes and links. The methods developed here included several analytical tools that allow for building a new GSG from a known one. Such theoretic methods are basically based on tensor (Kronecker) product of the adjacency matrices of the corresponding graphs. Consequently, they are easily implementable in

computer systems. On the other hand, we have designed a computer program which is able to find automatically regular bipartite GSGs. These graphs have the advantage that they are Ramanujan for a wide range of degrees in addition to their high synchronizability. We have also devised some analytic methods that allow to building new regular bipartite GSGs from known ones. In summary, we have found methods and algorithms for designing highly homogeneous networks, which are highly synchronizable and robust to random failures and intentional attacks. We also hope that the current work add an extra value to the approach to modeling networks based on tensor products of their adjacency matrices (Leskovec et al., 2009).

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Appendix. Adjacency matrices of bipartite regular GSGs

The adjacency matrices of the two regular bipartite graphs mentioned in the text are given as

$$\begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^{\mathbf{T}} & \mathbf{0} \end{pmatrix}$$

where we only shown the non-zero part **B** of the matrices.

1 1 0 0 1 1 1 1 1 0 0 1 1 0 1 1 1	0 1 1 1 1 0 1 1 0 1 1 1 1 1 1 0 1 1
0 1 1 1 1 1 1 0 1 1 1 1 0 0 1 0 1	1 1 1 1 1 0 0 1 1 1 1 1 1 1 1 0 0 1 1
1 1 1 1 0 1 1 1 1 0 1 1 0 1 0 1 0	0 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 0 0
1 1 1 1 0 1 1 1 0 1 0 1 1 1 1 0 0	1 1 0 1 1 1 1 0 0 0 1 1 1 1 1 1 1 1
0 0 0 1 1 1 1 1 1 1 1 1 1 1 0 0 1 1	0 0 1 1 1 1 1 0 1 1 1 1 1 1 1 1 0 1
0 1 1 0 1 1 0 1 1 1 1 1 1 0 1 1 1 0	1 0 1 1 1 1 1 1 1 1 0 1 1 1 0 1 1 0
1 0 1 0 0 1 1 1 1 1 1 1 1 1 0 1 0 1 1	1 0 1 1 1 1 1 1 0 1 1 1 1 1 1 1 0 1 0
1 1 1 1 1 0 1 0 1 1 0 1 1 1 0 1 0	1 1 1 1 1 1 0 1 1 1 1 0 0 1 1 0 1 1 1
0 1 1 1 1 0 1 1 1 0 1 0 1 1 1 1 0 1	1 1 1 1 1 1 1 1 1 1 1 0 1 0 0 1 1 1 0
1 0 1 1 1 1 0 0 1 1 1 1 1 1 1 0 0 1 1	1 1 1 1 1 1 0 1 0 0 1 0 1 1 1 1 1 1
1 1 0 0 1 1 1 0 1 1 1 1 1 1 1 1 0 0	1 1 1 1 1 1 1 1 0 1 1 0 0 1 0 1 1 1
1 1 0 1 1 1 1 1 0 0 1 1 0 0 1 1 1 1	1 1 0 0 0 1 1 1 1 1 1 1 1 1 1 0 1 1
1 0 0 1 0 1 1 1 0 1 1 0 1 1 1 1 1	1 1 1 0 0 1 1 1 0 1 1 1 1 1 1 1 0 1
1 0 1 1 1 1 0 1 0 1 0 1 1 1 1 0 1	1 1 1 1 0 1 1 1 1 1 0 1 0 0 1 1 1 1
0 1 1 1 1 1 0 1 1 0 0 0 1 1 1 1 1	1 0 0 1 0 1 1 1 1 1 1 0 1 1 1 1 1 1
1 1 1 1 0 0 0 0 1 0 1 1 1 1 1 1 1	0 1 0 0 1 1 1 1 1 1 1 1 1 1 0 1 1 1
1 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 1	1 1 1 0 1 1 1 1 1 1 1 1 0 0 1 1 0 1

Regular bipartite GSG with degree 12 | Regular bipartite GSG with degree 13

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