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Integer sequences from walks in graphs

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Abstract: We define numbers of the type $O_j(N) = N^0 - N^1 + N^2 - \ldots + N^{2j}$ and $E^j(N) = -N^0 + N^1 - N^2 + \ldots + N^{2j+1}$ ($j = 0, 1, 2, \ldots$) and the corresponding integer sequences. We prove that these integer sequences, e.g., $S_0(N) = O_0(N), O_1(N), \ldots, O_r(N), \ldots$ and $S_E(N) = E_0(N), E_1(N), \ldots, E_r(N), \ldots$ correspond to the number of odd and even walks in complete graphs K_N. We then prove that there is a unique family of graphs which have exactly the same sequence of odd walks between connected nodes and of even walks between pairs of nodes at distance two, respectively. These graphs are the *crown graphs*: $G_{2n} = K_2 \otimes K_n$.
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1 Introduction

Integer sequences arise from a few different sources, such as enumeration problems, number theory, game theory, physics, and so forth [1]. Enumeration problems in graph theory are an immense source of integer sequences as can be seen from an inspection of the *Online Encyclopedia of Integer Sequences* [2, 3]. In that particular case an investigation in the context of graph theory or combinatorics gives rise to a sequence of integers, which is then properly investigated. However, here we pose a different problem. For instance, let $N \in \mathbb{N}$ be a natural number ($N > 1$) and let us define the following numbers ($j = 0, 1, 2, \ldots$):

$$O_j(N) = N^0 - N^1 + N^2 - \ldots + N^{2j}, \tag{1.1}$$

$$E_j(N) = -N^0 + N^1 - N^2 + \ldots + N^{2j+1}. \tag{1.2}$$

Using $O_j(N)$ and $E_j(N)$ let us now introduce the following sequences:

$$S_O(N) = O_0(N), O_1(N), \ldots, O_r(N), \ldots \tag{1.3}$$

$$S_E(N) = E_0(N), E_1(N), \ldots, E_r(N), \ldots . \tag{1.4}$$

There are two $S_O(N)$ and three $S_E(N)$ sequences reported in the *Online Encyclopedia of Integer Sequences* (OEIS) [3]. They are:

Sequence	OEIS code	Current work
1, 3, 11, 43, 171, ...	A007583	$S_O(2)$
1, 7, 61, 547, 4921, ...	A066443	$S_O(3)$
0, 1, 5, 21, 85, 341, ...	A002450	0, $S_E(2)$ *
0, 2, 20, 182, 1640, ...	A125857	0, $S_E(3)$ *
0, 3, 51, 819, 13107, ...	A182512	0, $S_E(4)$ *

* The even sequences here do not include zero as in the OIES ones.

They appear, however, related to different mathematical objects. For instance, A007583, A002450 and A182512 appear associated to the formulae $(2^{2n+1} + 1) / 3$, $(4^n - 1) / 3$ and $(16^n - 1) / 5$, respectively. However, A066443 appears as the number of distinct walks of length $2n + 1$ along edges of a unit cube between two fixed adjacent vertices and A125857 represents the numbers whose base 9 representation is 22222222...2. For $N > 3$ no sequence of the type $S_O(N)$ is reported in the OEIS and the same is true for $S_E(N)$ when $N > 4$. So, the questions are: Is there a general framework in which all of these sequences can be grouped together? Are these sequences related to any property of a certain kind of graph?

Of course, these questions are not always possible to be answered. However, in the particular case that we can associate such sequences with a property of a type of graph we can yet ask another fundamental question. Given an integer sequence that represents a property of a given kind of graph, can we construct another family of graphs having the same sequence for this property? These questions are investigated here for the particular sequences (1.3) and (1.4). We find here that they correspond to the number of odd and even-length walks between pairs of nodes in complete graphs. We then prove that for each complete graph K_n there is a unique graph, not isomorphic to K_n, which has exactly the same sequence of walks. These graphs are the *bipartite double cover* of the complete graphs, which are known as *crown graphs.*

2 Preliminary results

For the sake of self-containment of this work we prove here a few basic results which are needed for proving the main result stated in the next section. We start by finding the general expressions for the numbers $O_j(N)$ and $E_j(N)$.

Lemma 2.1: The numbers $O_j(N)$ and $E_j(N)$ are positive integers given by the following formulae:

$$O_j(N) = \frac{N^{2j+1} + 1}{N+1}, \tag{2.1}$$

$$E_j(N) = \frac{N^{2j+2} - 1}{N+1}, \; j = 0, 1, \ldots . \tag{2.2}$$

Proof: The procedure is quite similar for both numbers, thus we are showing only it for $O_j(N)$. This number can be expressed as:

$$O_j(N) = \sum_{r=0}^{j} N^{2r} - \sum_{r=1}^{j} N^{2r-1} = \sum_{r=0}^{j} N^{2r} - N^{-1} \sum_{r=0}^{j} \left(N^{2r} - 1\right)$$
$$= \frac{1 - N^{2r+2}}{1 - N^2} - N^{-1}\left(\frac{1 - N^{2r+2}}{1 - N^2} - 1\right) \qquad (2.3)$$
$$= \frac{N^{2r+1} + 1}{N + 1}.$$

In order to show that the number is a positive integer it is enough to realize that

$$O_j(N) = N^{-1} + \sum_{r=0}^{j} N^{2r} - N^{-1} \sum_{r=0}^{j} N^{2r} > 0, \qquad (2.4)$$

which proves the result. In a similar way the proof for $E_j(N)$ is conducted. □

Now we prove the following result that relates the sequences $S_O(N)$ and $S_E(N)$ with the number of walks in complete graphs. Let us consider a simple graph without multiple links or self-loops $G = (V, E)$ with nodes (vertices) $v_i \in V$, $i = 1, \ldots, n$ and edges $\{v_i, v_j\} \in E$. A walk of length k is a sequence of (not necessarily distinct) nodes $v_0, v_1, \ldots, v_{k-1}, v_k$, such that for each $i = 1, 2, \ldots, k$ there is an edge from v_{i-1} to v_i. If $v_0 = v_k$ the walk is named a *closed walk*. A complete graph K_n is the graph having n nodes and every pair of nodes is connected by an edge.

Theorem 2.2: The sequence $S_O(N)$ and $S_E(N)$ give, respectively, the number of walks of odd and even lengths between pairs of nodes in a complete graph K_{n+1}.

Proof: Let $M^{2r+1}(p, q)$ be the number of odd walks of length $2r + 1$ between the nodes p and q in K_{N+1}. Then, it is known that

$$M^{2r+1}(p,q) = \sum_{j=1}^{N+1} \varphi_j(p)\varphi_j(q)\lambda_j^{2r+1}, \qquad (2.5)$$

where $\varphi_j(p)$ is the p^{th} entry of the orthonormalized eigenvector associated with the λ_j eigenvalue.

The principal eigenvalue of the adjacency matrix of K_{N+1} is N and its corresponding orthonormalized eigenvector is $\left(\sqrt{N+1}\right)^{-1}\mathbf{1}$, where $\mathbf{1}$ is an all-ones vector. The rest of the eigenvalues are equal to -1. Then,

$$M^{2r+1}(p,q) = \frac{1}{\sqrt{N+1}}\frac{1}{\sqrt{N+1}} N^{2r+1} + \sum_{j=2}^{N+1} \varphi_j(p)\varphi_j(q)(-1)^{2r+1}$$
$$= \frac{N^{2r+1}}{N+1} - \sum_{j=2}^{N+1} \varphi_j(p)\varphi_j(q) \qquad (2.6)$$
$$= \frac{N^{2r+1} + 1}{N + 1}.$$

In a similar way the result for the number of walks of even length $M^{2r}(p, q)$ between two nodes in K_{N+1} is proved. □

We now turn our attention to the graphs $G = H \otimes K_2$, which are known as the *bipartite double cover* of the graph H. Here $\otimes$ denotes the tensor (Kronecker product) of the adjacency matrices of the two graphs. In the next section we prove the main result of this work, which is related to the number of walks in the bipartite double covers of complete graphs: $G = K_2 \otimes K_N$. These graphs are known as *crown graphs*. We state now a few known or easy to prove properties of crown graphs [4–9]:

1) G_{2N} has $2N$ nodes, where N is the number of nodes in the original K_N;
2) G_{2N} is bipartite and regular with degree $k = N-1$;
3) The adjacency matrix of G_{2N} is

$$A(G_{2N}) = \begin{pmatrix} 0 & A(K_N) \\ A(K_N) & 0 \end{pmatrix};$$

4) The diameter of G_{2N} is 3 and its distance matrix is

$$D(G_{2N}) = \begin{pmatrix} 2A(K_N) & 3I + A(K_N) \\ 3I + A(K_N) & 2A(K_N) \end{pmatrix};$$

5) The number of walks of length r between the nodes i and j in the graph G_{2N} is given by:

$$M_{ij}^{r} = \begin{cases} \dfrac{k^{2r+1}+1}{k+1}, \text{ if } i \neq j, \ d_{ij} = 1 \text{ and } r \text{ is odd} \\[2ex] \dfrac{k^{2r}+1}{k+1}, \text{ if } i \neq j, \ d_{ij} = 2 \text{ and } r \text{ is even} \\[2ex] 0, \text{ if } i = j, \text{ and } r \text{ is odd} \\[2ex] \dfrac{k^{2r}+1}{k+1} + 1, \text{ if } i = j, \text{ and } r \text{ is even} \end{cases}$$

where $k = \dfrac{N}{2} - 1$ is the degree of a node in G_{2N}.

6) The spectrum of G_{2N} is $sp(G_{2N}) = \{[k]^1, [1]^{N-1}, [-1]^{N-1}, [-k]^1\}$;
7) The graphs G_{2N} are *distance-regular graphs*.

3 Main result

Now, we state the main result of this work. Let $S_O(N) = O_1(N), O_3(N), \ldots, O_{2r+1}(N), \ldots$ be the sequence of walks of odd length in a complete graph K_N. Then, by Lemma 2.1 and Theorem 2.2 we obtain that

$$O_j(N) = \frac{N^{2j+1}+1}{N+1}. \tag{3.1}$$

Theorem 3.1: Let G be a graph with adjacency matrix $A = (a_{ij})$. Then, for every edge $p - q$ of G the virtual power $a_{pq}^{(2j+1)} = O_j(N)$, for every $j \geq 1$, if and only if G is the bipartite double cover graph of a complete graph, $G = K_2 \otimes K_N$.

Proof: We divide the proof in several steps.

Proposition 3.2: Let G be a connected, bipartite almost complete graph with N nodes. Then the following hold:

(a) The spectral radius $\rho = N-1$;

(b) G is a regular graph with constant degree $k_p = N - 1$ for any node p.

Proposition 3.3: Let G be a connected, bipartite graph satisfying:

(c) G is a regular graph with constant degree $k_p = N - 1$ for any node p;

(d) For every edge $p - q$ we have $a_{pq}^{(2j+1)} = \dfrac{(N-1)^{2j+1}+1}{N}$ for $j = 1, 2$.

Then $G = K_2 \otimes K_N$.

We proceed with the proof of the Propositions.

Proof of Proposition 3.2: Let us write $B = A(K_N) = (b_{rs})$ for the adjacency matrix of K_N. Then for any edge $p - q$ in G we have

$$\rho = \lim_{j\to\infty} \sqrt[2j+1]{a_{pq}^{(2j+1)}} = \lim_{j\to\infty} \sqrt[2j+1]{b_{rs}^{(2j+1)}} = N-1. \tag{3.2}$$

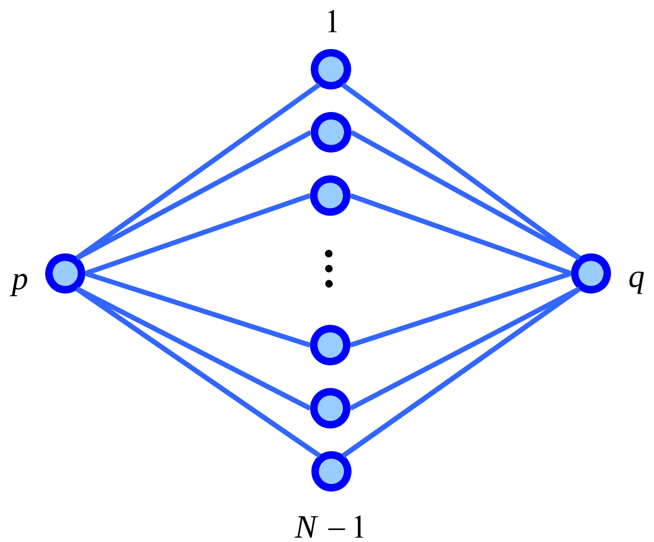
Let p be any vertex of G and choose an edge $p - q$. The matrix $\tilde{A} = \dfrac{1}{\rho}A$ is double stochastic, its (p, q) entry $\tilde{a}_{pq} = \dfrac{1}{N-1}$ measures the probability to go from node p to node q in the graph G. Therefore, $k_p = N - 1$. □

We need the following Lemma for the proof of *Proposition 3.3*.

Lemma 3.4: Let G be a bipartite graph satisfying (c) and (d) above. Then for any pair of nodes (p, q) in G the following holds:

$$a_{pq}^{(2)} \le N-2.$$

Proof: Assume otherwise that $a_{pq}^{(2)} > N-2$. Observe that $a_{pq}^{(2)} = \sum_s a_{ps}a_{sq} \le N-1$. Hence $a_{pq}^{(2)} = N-1$ and we get the following full subgraph H of G:



where all neighbouring vertices to p and q are 1, …, $N-1$. Let us calculate $a_{p1}^{(3)}$. Indeed, using hypothesis (d) we get

$$\left(N-1\right)\left(N-2\right)+1=a_{p1}^{(3)}=\sum_{j=2}^{N-1}a_{j1}^{(2)}+N-1, \tag{3.3}$$

which yields either one of the following two situations:

1) $a_{j1}^{(2)}=N-1$, for some $1\neq j$. Without loss of generality, for $j=2$ we have

$$N-1=a_{12}^{(2)}=\sum_s a_{1s}a_{s2} \tag{3.4}$$

Since there are no edges connecting any two of 1, …, $N-1$, because G is bipartite, there should exist vertices $2'$, …, $(N-2)'$ not in H, maybe not pairwise different, such that a_{1s}, $a_{s2}=1$, for each $s=2'$, …, $(N-2)'$. We can assume that $a_{1j'}$, $a_{j'2}=1$, for each $j=2$, …, $N-2$. and there are no paths of length two between 1 and $N-1$. Repeating the calculation done in (3.3) we get

$$\left(N-1\right)\left(N-2\right)+1=a_{p1}^{(3)}\leq\sum_{j=2}^{N-2}a_{j1}^{(2)}+N-1\leq\left(N-1\right)\left(N-3\right)+N-1, \tag{3.5}$$

which is a contradiction.

2) For $i\neq j$ in the set 1, …, $N-1$, we have $a_{ij}^{(2)}=N-2$. Then

$$\frac{\left(N-1\right)^5+1}{N}=a_{p1}^{(5)}\geq\sum_{j=1}^{N-1}a_{pj}^{(3)}a_{j1}^{(2)}\geq\frac{\left(N-1\right)^3+1}{N}\left[\left(N-1\right)\left(N-2\right)+N-1\right], \tag{3.6}$$

which also yields a contradiction. □

Proof of Proposition 3.3: We shall now describe the structure of the matrix A^2. For that purpose we shall calculate all virtual powers $a_{ij}^{(2)}$.

First, for each vertex p we have $a_{pp}^{(2)}=k_p=N-1$. Let p, q be different vertices in G. By the Lemma above, $a_{pq}^{(2)}\leq N-2$. Assume that $a_{pq}^{(2)}\neq 0$ and take any vertex j with edges $p-j-q$. Then for the neighbours $1,\ldots,N-1$ of j we get, using Proposition 3.2,

$$\left(N-1\right)\left(N-2\right)+1=a_{pj}^{(3)}\leq\sum_{p\neq j=1}^{N-1}a_{pi}^{(2)}+N-1\leq\left(N-2\right)^2+N-1, \tag{3.7}$$

which implies equality $a_{pq}^{(2)}=N-2$.

We observe that the number of vertices of the graph G is $n=2N$. Indeed, since G is bipartite, we get two non-empty classes V_1, V_2 of totally disconnected vertices in the set 1, …, n. Let p, q, s be three different vertices in V_1 and $p-j-q-i$ edges. Then

$$a_{pq}^{(2)}=N-2=a_{qs}^{(2)}. \tag{3.8}$$

Since this number is not zero, there is a vertex k in V_2 which is connected to p and s, and $a_{ps}^{(2)}\neq 0$. Therefore A^2 has $a_{pq}^{(2)}=N-2$ or 0 depending if p, q belong or not to the same class V_i. Hence each of V_1 and V_2 have the same cardinality N.

Moreover it is clear that the adjacency matrix A has entries $a_{pq}=0$ or 1 depending if p, q are different and in distinct classes or belong to the same class V_i, respectively. As desired, it follows that $A=A(K_2)\otimes A(K_N)$. □

4 Closing remarks

We have introduced two new integer sequences which we prove correspond to the sequences of even and odd walks in complete graphs K_n. The fundamental question posted here is whether there are other graphs with exactly the same sequence of walks as those of the complete graphs. We have proved that the answer is positive and the graphs having such sequences are unique. They correspond to the bipartite double covers of the complete graphs, known as crown graphs: $G_{2n} = K_2 \otimes K_n$.

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